Math 3140 - Assignment 3

Due September 11, 2026

(1) The **center** $Z(G)$ of a group $(G, \cdot, ^{-1}, 1)$ is the set of elements in G that commute with every element in G, that is,

$$Z(G) := \{a \in G \ : \ ax = xa \text{ for all } x \in G\}.$$

Show that $Z(G)$ is a subgroup of G.

Solution: Check the subgroup properties:
(a) $1 \in Z(G)$ since $1x = x = x1$ for all $x \in G$
(b) For all $a, b \in Z(G)$ and $x \in G$, we have $abx = axb = xab$ and hence $ab \in Z(G)$.
(c) For all $a \in Z(G)$ and $x \in G$, we have $ax = xa$. Multiplying this equation with a^{-1} left and right yields $xa^{-1} = a^{-1}x$. Hence $a^{-1} \in Z(G)$.

(2) Determine the center of $\mathrm{GL}(2, \mathbb{R})$.

Hint: For $1 \leq i, j \leq 2$, let E_{ij} be the 2×2 matrix whose ij-entry is 1 and all other entries are 0. This is not invertible but their sum with the identity matrix $I + E_{ij}$ is.
Note that $A(I + E_{ij}) = (I + E_{ij})A$ iff $AE_{ij} = E_{ij}A$. Check the latter equations to determine conditions on a, b, c, d such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(\mathrm{GL}(2, \mathbb{R}))$.

Solution: Following the hint consider $A \cdot E_{12} = E_{12} \cdot A$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

Equality yields that $a = d$ and $c = 0$.

Dually, multiplying A and E_{21} yields that $a = d$ and $b = 0$.

So any A in $Z(\mathrm{GL}(2, \mathbb{R}))$ is a multiple of the identity matrix aI_2 for $a \in \mathbb{R}$. Conversely every aI_2 clearly commutes with all matrices. So

$$Z(\mathrm{GL}(2, \mathbb{R})) = \{aI_2 \ : \ a \in \mathbb{R} \setminus \{0\}\}.$$

(3) Prove that every group of even order has an element of order 2.

Hint: Consider the pairs $\{x, x^{-1}\}$ in the group.

Solution: $A := \{x \in G \ : \ x^{-1} \neq x\}$ can be partitioned into pairs x, x^{-1}, hence $|A|$ is even.

Now G is the disjoint union $1 \cup A \cup \{x \in G \ : \ x$ has order $2\}$. Hence

$$|G| \equiv 1 + |\{x \in G \ : \ x \text{ has order } 2\}| \mod 2$$

yields that a group G of even order has an odd number of elements of order 2.

(4) Which of the following groups are cyclic? For those that are, list all their generators. For those that are not, explain why.

$A = (\mathbb{Q}, +)$
$B = (\mathbb{Z}_{12}, +)$
$C = (\mathbb{Z}_7^*, \cdot)$
$D = \{\pi^z \ : \ z \in \mathbb{Z}\}$ under multiplication
$E = \mathbb{Z}^2 = \{(a, b) \ : \ a, b \in \mathbb{Z}\}$ under addition

Solution:

A is not cyclic: For any $\frac{a}{b} \in \mathbb{Q}$ we have $\langle \frac{a}{b} \rangle = \{\frac{az}{b} \ : \ z \in \mathbb{Z}\} \neq \mathbb{Q}$ since it does not contain e.g. $\frac{a}{2b}$.

B is cyclic with generators $1, 5, 7, 11$ (the classes of elements whose gcd with 12 is 1).

C is cyclic with generators $3, 5$

D is cyclic with generators π, π^{-1}

E is not cyclic: For any $(a, b) \in \mathbb{Z}^2$ we have $\langle (a, b) \rangle = \{z(a, b) \ : \ z \in \mathbb{Z}\} \neq \mathbb{Z}^2$ since it is contained in a 1-dimensional subspace of $\mathbb{R}^2$.

(5) How many subgroups does $(\mathbb{Z}_{12}, +)$ have? List a generator for each subgroup. Draw a diagram showing the containments between the subgroups.

Solution: The subgroup lattice looks like the lattice of divisors of 12 upside down. Something like

$$
\begin{tikzcd}
 & \langle 1\rangle \arrow[dl, dash] \arrow[dr, dash] & & \\
\langle 3\rangle \arrow[dr, dash] & & \langle 2\rangle \arrow[dl, dash] \arrow[dr, dash] & \\
 & \langle 6\rangle \arrow[dr, dash] & & \langle 4\rangle \arrow[dl, dash] \\
 & & \langle 0\rangle &
\end{tikzcd}
$$

There are 6 subgroups (same as the number of divisors of 12).