

# Math 3140 - Assignment 3

Due September 11, 2026

- (1) The **center**  $Z(G)$  of a group  $(G, \cdot, ^{-1}, 1)$  is the set of elements in  $G$  that commute with every element in  $G$ , that is,

$$Z(G) := \{a \in G : ax = xa \text{ for all } x \in G\}.$$

Show that  $Z(G)$  is a subgroup of  $G$ .

**Solution:** Check the subgroup properties:

- (a)  $1 \in Z(G)$  since  $1x = x = x1$  for all  $x \in G$
- (b) For all  $a, b \in Z(G)$  and  $x \in G$ , we have  $abx = axb = xab$  and hence  $ab \in Z(G)$ .
- (c) For all  $a \in Z(G)$  and  $x \in G$ , we have  $ax = xa$ . Multiplying this equation with  $a^{-1}$  left and right yields  $xa^{-1} = a^{-1}x$ . Hence  $a^{-1} \in Z(G)$ .

- (2) Determine the center of  $\text{GL}(2, \mathbb{R})$ .

Hint: For  $1 \leq i, j \leq 2$ , let  $E_{ij}$  be the  $2 \times 2$  matrix whose  $ij$ -entry is 1 and all other entries are 0. This is not invertible but their sum with the identity matrix  $I + E_{ij}$  is.

Note that  $A(I + E_{ij}) = (I + E_{ij})A$  iff  $AE_{ij} = E_{ij}A$ . Check the latter equations to determine conditions on  $a, b, c, d$  such that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(\text{GL}(2, \mathbb{R})).$$

**Solution:** Following the hint consider  $A \cdot E_{12} = E_{12} \cdot A$ ,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}.$$

Equality yields that  $a = d$  and  $c = 0$ .

Dually, multiplying  $A$  and  $E_{21}$  yields that  $a = d$  and  $b = 0$ .

So any  $A$  in  $Z(\text{GL}(2, \mathbb{R}))$  is a multiple of the identity matrix  $aI_2$  for  $a \in \mathbb{R}$ . Conversely every  $aI_2$  clearly commutes with all matrices. So

$$Z(\text{GL}(2, \mathbb{R})) = \{aI_2 : a \in \mathbb{R} \setminus \{0\}\}.$$

- (3) Prove that every group of even order has an element of order 2.

Hint: Consider the pairs  $\{x, x^{-1}\}$  in the group.

**Solution:**  $A := \{x \in G : x^{-1} \neq x\}$  can be partitioned into pairs  $x, x^{-1}$ , hence  $|A|$  is even.

Now  $G$  is the disjoint union  $1 \cup A \cup \{x \in G : x \text{ has order } 2\}$ . Hence

$$|G| \equiv 1 + |\{x \in G : x \text{ has order } 2\}| \pmod{2}$$

yields that a group  $G$  of even order has an odd number of elements of order 2.

- (4) Which of the following groups are cyclic? For those that are, list all their generators. For those that are not, explain why.

$$A = (\mathbb{Q}, +)$$

$$B = (\mathbb{Z}_{12}, +)$$

$$C = (\mathbb{Z}_7^*, \cdot)$$

$$D = \{\pi^z : z \in \mathbb{Z}\} \text{ under multiplication}$$

$$E = \mathbb{Z}^2 = \{(a, b) : a, b \in \mathbb{Z}\} \text{ under addition}$$

**Solution:**

$A$  is not cyclic: For any  $\frac{a}{b} \in \mathbb{Q}$  we have  $\langle \frac{a}{b} \rangle = \{\frac{az}{b} : z \in \mathbb{Z}\} \neq \mathbb{Q}$  since it does not contain e.g.  $\frac{a}{2b}$ .

$B$  is cyclic with generators 1, 5, 7, 11 (the classes of elements whose gcd with 12 is 1).

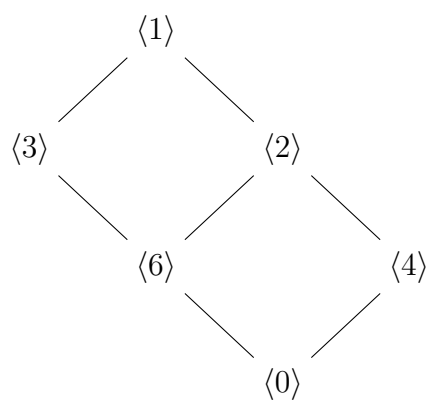
$C$  is cyclic with generators 3, 5

$D$  is cyclic with generators  $\pi, \pi^{-1}$

$E$  is not cyclic: For any  $(a, b) \in \mathbb{Z}^2$  we have  $\langle (a, b) \rangle = \{z(a, b) : z \in \mathbb{Z}\} \neq \mathbb{Z}^2$  since it is contained in a 1-dimensional subspace of  $\mathbb{R}^2$ .

- (5) How many subgroups does  $(\mathbb{Z}_{12}, +)$  have? List a generator for each subgroup. Draw a diagram showing the containments between the subgroups.

**Solution:** The subgroup lattice looks like the lattice of divisors of 12 upside down. Something like



There are 6 subgroups (same as the number of divisors of 12).