Math 3140 - Assignment 3

Due September 11, 2026

(1) The **center** $Z(G)$ of a group G is the set of elements in G that commute with every element in G, that is,

$$Z(G) := \{a \in G \ : \ ax = xa \text{ for all } x \in G\}.$$

Show that $Z(G)$ is a subgroup of G.

(2) Determine the center of $\mathrm{GL}(2, \mathbb{R})$.

Hint: For $1 \leq i, j \leq 2$, let E_{ij} be the 2×2 matrix whose ij-entry is 1 and all other entries are 0. This is not invertible but their sum with the identity matrix $I + E_{ij}$ is.

Note that $A(I + E_{ij}) = (I + E_{ij})A$ iff $AE_{ij} = E_{ij}A$. Check the latter equations to determine conditions on a, b, c, d such that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Z(\mathrm{GL}(2, \mathbb{R}))$.

(3) Prove that every group of even order has an element of order 2.

Hint: Consider the pairs $\{x, x^{-1}\}$ in the group.

(4) Which of the following groups are cyclic? For those that are, list all their generators. For those that are not, explain why.

$A = (\mathbb{Q}, +)$
$B = (\mathbb{Z}_{12}, +)$
$C = (\mathbb{Z}_7^*, \cdot)$
$D = \{\pi^z \ : \ z \in \mathbb{Z}\}$ under multiplication
$E = \mathbb{Z}^2 = \{(a, b) \ : \ a, b \in \mathbb{Z}\}$ under addition

(5) How many subgroups does $(\mathbb{Z}_{12}, +)$ have? List a generator for each subgroup. Draw a diagram showing the containments between the subgroups.