Math 2135 - Practice Midterm 2

Except for problem 1, give full justifications and computations for all your answers!

(1) True or false?

(a) _____ The dimensions of $\operatorname{Col} A$ and $\operatorname{Row} A$ are the same for every matrix A.

Solution: True, $\dim \operatorname{Col} A =$ number of pivots in the row echelon form of $A = \dim \operatorname{Row} A$. □

(b) _____ For $n \in \mathbb{N}$, a linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is bijective iff the kernel of f is trivial.

Solution: True. Assume f has standard matrix $A \in \mathbb{R}^{n\times n}$. Then the kernel of f is $\operatorname{Null} A$ which is trivial iff A has a pivot in every column. Since A is square, this is equivalent to the columns of A spanning $\mathbb{R}^n$, which is equivalent to $f(\mathbb{R}^n) = \mathbb{R}^n$, i.e., f is surjective, i.e. f is bijective. □

(2) (a) List the conditions under which a function $f\colon V \to W$ for vector spaces V, W over $\mathbb{R}$ is linear.

(b) Let $A, B, C \in \mathbb{R}^{n\times n}$. Assume that A is invertible. Show that if $AB = AC$, then $B = C$.

Solution: (b) Assume $AB = AC$. Multiply this equation with A^{-1} on the left to get

$$A^{-1}AB = A^{-1}AC.$$

Since A^{-1} and A cancel, this yields $B = C$. □

(3) Let $T\colon P_2 \to \mathbb{R}, p \mapsto p(3)$, be the map that evaluates a polynomial $p(t)$ at $t = 3$.

(a) Show that T is linear.

(b) Determine the kernel of T, that is, $\ker T = \{p \in P_2 : T(p) = 0\}$, and the image of T, that is, $T(P_2)$.

(c) Is T injective, surjective, bijective?

Solution:

(a) For linearity, let $p, q \in P_2$. Their sum $p + q$ is the polynomial that maps t to $p(t) + q(t)$. So

$$T(p+q) = (p+q)(3) = p(3) + q(3) = T(p) + T(q).$$

Further let $c \in \mathbb{R}$. Then cp maps t to $cp(t)$. So

$$T(cp) = (cp)(3) = cp(3) = cT(p).$$

Hence T is linear.

(b) The kernel of T, $\ker T$, consists of all the polynomials that evaluate to 0 at 3, that is,

$$\ker T = \{(t-3)q : q \in P_1\}.$$

The range of T, $T(P_2)$, is $\mathbb{R}$. For every $b \in \mathbb{R}$, there exists a polynomial $p \in P_2$ that is mapped to b. Choose for example the constant polynomial $p(t) = b$.

(c) Since the kernel of T is non-trivial, T is not injective.
Since the range of T is equal to its codomain, T is surjective.
T is not bijective since it is not injective.

□

(4) Let $B = (b_1, b_2)$ with $b_1 = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $C = (\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix})$ be bases of a subspace H of $\mathbb{R}^3$.

(a) Compute the coordinates $[b_1]_C$ and $[b_2]_C$.
(b) What is the change of coordinate matrix $P_{C \leftarrow B}$?
(c) What is the change of coordinate matrix $P_{B \leftarrow C}$?

Solution: (a) Solve the linear system

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}$$

to obtain $x_1 = 3, x_2 = -4$. So $[b_1]_C = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

Similarly we get $[b_2]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Since the columns of $P_{C \leftarrow B}$ are just the coordinate tuples $[b_i]_C$, we see

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$$

(c)

$$P_{B \leftarrow C} = P_{C \leftarrow B}^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$$

Alternatively we could also get $P_{B \leftarrow C}$ from its columns $[c_i]_B$.

□

(5) Let $C = (1 + t, t + t^2, 1 + t^2)$ be a basis for P_2. Compute the coordinates $[p]_C$ for $p = 2 + t^2$.

Solution: Solve

$$c_1(1 + t) + c_2(t + t^2) + c_3(1 + t^2) = 2 + t^2.$$

Comparing the coefficients on both sides of this equation yields

$$\begin{aligned} c_1 + c_3 &= 2 \qquad \text{(constant part)} \\ c_1 + c_2 &= 0 \qquad \text{(multiples of } t\text{)} \\ c_2 + c_3 &= 1 \qquad \text{(multiples of } t^2\text{)} \end{aligned}$$

Solving that system of linear equations yields $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}, c_3 = \frac{3}{2}$. So $[u]_B = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$. $\square$