# Math 2135 - Assignment 10

Due November 9, 2026

(1) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

**Solution:**
Expand $\det A$ down the first column:

$$\det A = 0 \cdot \det A_{11} - 5 \cdot \det A_{21} + 0 \cdot \det A_{31} = -5 \cdot \det \begin{bmatrix} 1 & -3 \\ -3 & -4 \end{bmatrix} = -5(1(-4) - (-3)(-3)) = 65$$

Expand $\det B$ across 3rd row:

$$\det B = 2 \cdot \det B_{13} = 2 \cdot \det \begin{bmatrix} 0 & -3 & 0 \\ 1 & 5 & 1 \\ 1 & -2 & 5 \end{bmatrix}$$

Expand across 1st row:

$$\det B_{13} = -1(-3) \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 3 \cdot (1 \cdot 5 - 1 \cdot 1) = 12$$

So $\det B = 2 \cdot 12 = 24$. □

(2) **Rule of Sarrus (basketweaving) for the determinant of $3 \times 3$-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

**Solution:**

$$\begin{aligned} \det A &= a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13} \\ &= a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{aligned}$$

□

(3) Give two $3 \times 3$-matrices with determinant 6. (Hint: triangular matrices.)

**Solution:**
Any triangular or diagonal matrix whose diagonal elements multiply to 6 will do, e.g.,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

□

(4) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

**Solution:**
Interchanging 2 rows changes the sign of the determinant:

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = -\det A$$

□

(b) How does multiplying one row by a scalar effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

(c) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix}$.

**Solution:**
Adding a multiple of the first row to another does not change the determinant:

$$\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix} = a(d + rb) - b(c + ra) = ad - bc = \det A$$

□

(5) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} u & v \\ w & x \end{bmatrix}$. Show

$$\det(AB) = \det(A) \det(B).$$

**Solution:**

$$AB = \begin{bmatrix} au + bw & av + bx \\ cu + dw & cv + dx \end{bmatrix}$$

$$\det AB = (au + bw)(cv + dx) - (av + bx)(cu + dw) = \cdots = \det A \cdot \det B$$

□

(6) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} \det A &= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} && \text{factoring 3 from the first row} \\ &= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{bmatrix} && \text{subtracting multiples of the first row from the others} \\ &= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix} && \text{adding 5 times the second row to the third} \\ &= 3 \cdot 1 \cdot 1 \cdot (-8) = -24. \end{aligned}$$

$$\begin{aligned} \det B &= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{bmatrix} \\ &= -\det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{bmatrix} && \text{flipped row 3 and 4} \\ &= -1 \cdot 1 \cdot 1 \cdot 10 = -10. \end{aligned}$$

□