Math 2135 - Assignment 9

Due October 30, 2026

(1) (a) A 25×35 matrix A has 20 pivots. Find $\dim \operatorname{Null} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.

Solution:
The number of pivots, $\dim \operatorname{Row} A$, $\dim \operatorname{Col} A$, and the rank are equal. So

$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A = 20.$$

By the rank theorem, $\dim \operatorname{Null} A + \operatorname{rank} A = 35$. Thus

$$\dim \operatorname{Null} A = 35 - 20 = 15.$$

□

(b) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of Nul A?

Solution:
The rank of a matrix is the number of its pivot elements, which is at most the number of its rows and at most the number of its columns. So $\operatorname{rank} A \leq \max(3, 4) = 3$. Since the largest possible rank is 3, the smallest number of free variables in $Ax = 0$ is 1. So the dimension of Nul A is 1 or larger. □

(c) If the nullspace of a 4×6-matrix B has dimension 4, what is the dimension of the row space of B?

Solution:
dim Nul A + dim Row A = the number of columns of A
So dim Row $A = 6 - 4 = 2$. □

(2) Show that $A \in \mathbb{R}^{n \times n}$ is invertible iff $\operatorname{rank} A = n$.

Solution:
$A \in \mathbb{R}^{n \times n}$ is invertible
iff the columns of A form a basis of $\mathbb{R}^n$ by the Inverse Matrix Theorem
iff $\operatorname{rank} A = n$ by the definition of rank as the dimension of Col A. □

(3) Let $B = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix})$ and $C = (\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.

(a) Find the change of coordinates matrix $P_{E \leftarrow B}$ for $f: [u]_B \mapsto [u]_E$.
(b) Find the change of coordinates matrix $P_{C \leftarrow E}$ for $g: [u]_E \mapsto [u]_C$.
(c) Find the change of coordinates matrix $P_{C \leftarrow B}$ for $h: [u]_B \mapsto [u]_C$.
Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$. So $P_{C \leftarrow B}$ is the product of $P_{C \leftarrow E}$ and $P_{E \leftarrow B}$. Careful about the order of the product.

Solution:
Let E be the standard basis of $\mathbb{R}^2$.

(a) How to compute E-coordinates from B-coordinates? The standard matrix for f is

$$P_{B\leftarrow E} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Note that the columns are exactly the vectors of B. Changing coordinates from any B to the standard basis E is easy.

(b) How to compute C-coordinates from E-coordinates? The standard matrix for g is

$$P_{E\leftarrow C} = P_{C\leftarrow E}^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Note that the matrix is the inverse of the matrix whose columns are the vectors of C. For changing coordinates from the standard basis E to a basis C you need to solve a linear system or find the inverse.

(c) How to compute C-coordinates from B-coordinates? First go from B-coordinates to E-coordinates and then to C-coordinates. The matrix for $h = g \circ f$ is

$$P_{C\leftarrow B} = P_{C\leftarrow E}P_{E\leftarrow B} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$$

□

(4) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:

(a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.

(b) Give the matrix $t_{B\leftarrow B}$ for the reflection with respect to the coordinate system determined by B.

(c) Use change of coordinate matrices to compute the standard matrix $t_{E\leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

Solution:

(a) Pick $B = (\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix})$ with the first vector b_1 on the line $3x + y = 0$, the second b_2 orthogonal. Then $t(b_1) = b_1, t(b_2) = -b_2$.

(b)

$$t_{B\leftarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

looks like the standard matrix for the reflection on the x-axis.

(c) To get $t_{E\leftarrow E}$ from $[t]_{B,B}$ we need to multiply with change of coordinate matrices,

$$t_{E\leftarrow E} = P_{B\leftarrow E}t_{B\leftarrow B}P_{E\leftarrow B} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 \\ -3 & -1 \end{bmatrix}\frac{1}{10}\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -8 & -6 \\ -6 & 8 \end{bmatrix}$$

□

(5) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?

Solution:
e_3 remains fixed, e_1, e_2 rotate like in $\mathbb{R}^2$, i.e.,

$$A = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 & 0 \\ \sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

(6) Let V, W be vector spaces over $\mathbb{R}$ with zero vectors $0_V, 0_W$, respectively. Let $f\colon V \to W$ be linear. Show
(a) $f(0_V) = 0_W$,
(b) the kernel $\ker f := \{v \in V \ : \ f(v) = 0_W\}$ of f is a subspace of V.

Solution:
(a) By linearity $f(0_V) = f(0_V + 0_V) = f(0_V) + f(0_V)$. Subtracting $f(0_V)$ from both sides yields $0_W = f(0_v)$.
(b) $\ker f$ contains 0_V by (a).

Show $\ker f$ is closed under addition: Let $u, v \in \ker f$, that is, $f(u) = 0, f(v) = 0$. By linearity $f(u + v) = f(u) + f(v) = 0 + 0 = 0$. Hence $u + v \in \ker f$.

Show $\ker f$ is closed under scalar multiples: Let $c \in \mathbb{R}$ and $v \in \ker f$, that is, $f(v) = 0$. By linearity $f(cv) = cf(v) = c0 = 0$. Hence $cv \in \ker f$. □

(7) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.
(a) Find the matrix $d_{B\leftarrow B}$ for the derivation map $d\colon P_3 \to P_3,\ p \mapsto p'$.
(b) Use $d_{B\leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.

Solution:

(a) Compute coordinates of the derivatives $d(b_i)$ for the basis vectors in B to get

$$d_{B\leftarrow B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Then $[p']_B = d_{B\leftarrow B}[p]_B = (2, 0, 3, 0)$ and $p' = 2 + 3x^2$.

□

(8) Problem 7 continued:
(a) Determine the kernel and the image of $d\colon P_3 \to P_3, p \to p'$.
(b) Is d injective, surjective, bijective?

Solution:

(a) $p \in \ker d$ iff $p' = 0$ iff p is a constant polynomial iff p has degree ≤ 0. Hence

$$\ker d = P_0.$$

For $p \in P_3$ we see that p' has degree 2, hence $p' \in P_2$. So $d(P_3) \subseteq P_2$.
We claim that conversely every $q = a_0 + a_1 x + a_2 x^2 \in P_2$ is in $d(P_3)$, that is the derivative of a polynomial p of degree 3. Note that $p = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3$

satisfies $p' = q$. Thus

$$d(P_3) = P_2.$$

(b) d is not injective since $\ker d \neq 0$.

d is not surjective since $d(P_3) = P_2 \neq P_3$.

Hence d is not bijective.

Alternatively for (b) one can also compute the nullspace of the matrix $d_{B \leftarrow B}$ to see that $\ker d$ is not trivial and its column space to see that $d(P_3) \neq P_3$. But for (a) you'd need to translate $\operatorname{Null} d_{B \leftarrow B}$ back to P_0 and $\operatorname{Col} d_{B \leftarrow B}$ back to P_2. □