Math 2135 - Assignment 9

Due October 30, 2026

(1) (a) A 25×35 matrix A has 20 pivots. Find $\dim \operatorname{Null} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.
   (b) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of $\operatorname{Nul} A$?
   (c) If the nullspace of a 4×6-matrix B has dimension 4, what is the dimension of the row space of B?

(2) Show that $A \in \mathbb{R}^{n \times n}$ is invertible iff $\operatorname{rank} A = n$.

(3) Let $B = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix})$ and $C = (\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.
   (a) Find the change of coordinates matrix $P_{E \leftarrow B}$ for $f: [u]_B \mapsto [u]_E$.
   (b) Find the change of coordinates matrix $P_{C \leftarrow E}$ for $g: [u]_E \mapsto [u]_C$.
   (c) Find the change of coordinates matrix $P_{C \leftarrow B}$ for $h: [u]_B \mapsto [u]_C$.
   Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$. So $P_{C \leftarrow B}$ is the product of $P_{C \leftarrow E}$ and $P_{E \leftarrow B}$. Careful about the order of the product.

(4) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:
   (a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.
   (b) Give the matrix $t_{B \leftarrow B}$ for the reflection with respect to the coordinate system determined by B.
   (c) Use change of coordinate matrices to compute the standard matrix $t_{E \leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

(5) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
   Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?

(6) Let V, W be vector spaces over $\mathbb{R}$ with zero vectors $0_V, 0_W$, respectively. Let $f: V \to W$ be linear. Show
   (a) $f(0_V) = 0_W$,
   (b) the kernel $\ker f := \{v \in V \ : \ f(v) = 0_W\}$ of f is a subspace of V.

(7) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.
   (a) Find the matrix $d_{B \leftarrow B}$ for the derivation map $d: P_3 \to P_3,\ p \mapsto p'$.
   (b) Use $d_{B \leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.

(8) Problem 7 continued:
   (a) Determine the kernel and the image of $d: P_3 \to P_3, p \to p'$.
   (b) Is d injective, surjective, bijective?