# Math 2135 - Assignment 7

Due October 16, 2026

(1) Explain why the following are not subspaces of $\mathbb{R}^2$. Give explicit counter examples for subspace properties that are not satisfied.

(a) $U = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, x \geq 0\}$

(b) $V = \mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of all integers)

(c) $W = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, |x| = |y|\}$ ($|x|$ denotes the absolute value of $x$).

**Solution:**

(a) Not closed under scalar multiples, e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U$ but $(-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin U$

(b) Not closed under scalar multiples, e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$ but $\sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin V$

(c) Not closed under addition, e.g. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in W$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin W$

□

(2) Let $A \in \mathbb{R}^{m \times n}$. Prove that $\mathrm{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

**Solution:**

We show the 3 conditions for being a subspace.

(a) The zero vector is clearly in $\mathrm{Nul}(A)$ since $A\mathbf{0} = \mathbf{0}$.

(b) Let $\mathbf{u}$ and $\mathbf{w}$ be arbitary vectors in $\mathrm{Nul}(A)$. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. We show that $\mathbf{u} + \mathbf{w}$ is in $\mathrm{Nul}(A)$.

$$A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So $\mathbf{u} + \mathbf{w}$ is in $\mathrm{Nul}(A)$.

(c) Let $r \in \mathbb{R}$. Then

$$A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}.$$

Hence $r\mathbf{u}$ is in $\mathrm{Nul}(A)$.

□

(3) Which of the following are subspaces of the vector space $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$? Check all subspace properties or give one that is not satisfied.

(a) $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 1\}$

(b) $\{f : \mathbb{R} \to \mathbb{R} \mid f(3) = 0\}$

(c) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$

**Solution:**

(a) No subspace since it does not contain the zero vector, i.e., the constant 0-function.

(b) Subspace since (1) contains the constant 0-function, (2) is closed under addition [for functions $f, g$ with $f(1) = 0$ and $g(1) = 0$, also $(f+g)(1) = 0+0 = 0$], (3) is closed under scalar multiples [if $c \in \mathbb{R}$ and $f(1) = 0$, then also $(cf)(0) = c0 = 0$].
(c) Subspace since (1) the constant 0-function is continuous, (2) the sum of continuous functions is continuous, (3) any scalar multiple of a continuos function is continuous.

□

(4) Explain whether the following are true or false (give counter examples if possible):
(a) Every vector space is a subspace of itself.
(b) Each plane in $\mathbb{R}^3$ is a subspace.
(c) Let $U$ be a subspace of a vector space $V$. Any linear combination of vectors of $U$ is also in $V$.
(d) Let $v_1, \ldots, v_n$ be in a vector space $V$. Then $\mathrm{Span}(v_1, \ldots, v_n)$ is the smallest subspace of $V$ containing $v_1, \ldots, v_n$.

**Solution:**

(a) True, since it contains zero vector, is closed under addition and scalar multiples.
(b) False, e.g., the plane $z = 1$ does not contain the zero vector.
(c) True, since $U$ is closed under under addition and scalar multiples by definition, $U$ is also closed under linear combinations.
(d) True, by (c) every subspace of $V$ containing $v_1, \ldots, v_n$ also contains $\mathrm{Span}(v_1, \ldots, v_n)$ which is a subspace by a previous HW-problem. So $\mathrm{Span}(v_1, \ldots, v_n)$ is the smallest subspace of $V$ containing $v_1, \ldots, v_n$.

□

(5) Are the vectors $\mathbf{v}_0 = 1$, $\mathbf{v}_1 = t$, $\mathbf{v}_2 = t^2$ in the vector space $\mathbb{R}^{\mathbb{R}} := \{f \colon \mathbb{R} \to \mathbb{R}\}$ linearly independent?

**Solution:**

Yes, for real numbers $x_0, x_1, x_2$ the polynomial $x_0 + x_1 t + x_2 t^2$ is equal to the constant 0-function iff $x_0 = x_1 = x_2 = 0$.

Alternatively, consider the equation

$$x_0 + x_1 t + x_2 t^2 = 0$$

at distinct values for $t$, e.g., $t = 0, 1, 2$ to obtain the linear system

$$x_0 + 0x_1 + 0^2 x_2 = 0$$
$$x_0 + 1x_1 + 1^2 x_2 = 0$$
$$x_0 + 2x_1 + 2^2 x_2 = 0$$

This only has the trivial solution $x_0 = x_1 = x_2 = 0$. So $1, t, t^2$ are linearly independent. □

(6) Which of the following are bases of $\mathbb{R}^3$? Why or why not?

$$A = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}), B = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\4\end{bmatrix}), C = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix})$$

**Solution:**
$A$ is not a basis because 2 vectors can at most span a plane but not all of $\mathbb{R}^3$.

To check whether $B$ is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & -1 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Since we have a 0-row, the vectors in $B$ do not span $\mathbb{R}^3$. Hence $B$ is not a basis.

To check whether $C$ is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 8 \end{bmatrix}$$

The echelon form has no 0-row. So $C$ spans $\mathbb{R}^3$. Further we see from the echelon form that $C$ is linearly independent. So $C$ is a basis. □

(7) Give a basis for $\mathrm{Nul}(A)$ and a basis for $\mathrm{Col}(A)$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 1 & -4 & -1 & 0 \\ -2 & 6 & 2 & -3 \end{bmatrix}$$

**Solution:**
Nul $A$ is the solution set of $Ax = 0$. So we row reduce $A$

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 1 & -4 & -1 & 0 \\ -2 & 6 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & -2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to get the solution $x_4 = t, x_3 = s$ (both free), $x_2 = -\frac{3}{2}t, x_1 = s - 6t$. So

$$x = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \text{ and } \mathrm{Nul}\, A \text{ has basis } \left( \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

For a basis of the column space Col$A$ we pick the pivot columns of $A$, i.e., the first and second column. So Col$A$ has basis $\left( \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} \right)$. □

(8) Give 2 different bases for

$$H = \mathrm{Span}\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \}$$

**Solution:**
Row reduction yields

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the first 2 columns of the original matrix are pivot columns and form a basis of $H$.

$$B_1 = (\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix})$$

Recall that the order of basis vectors is important (coordinates!). So by flipping the vectors we get a different basis

$$B_2 = (\begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix})$$

Also note that any 2 columns of the matrix are linearly independent. So any 2 distinct vectors in any order out of $\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix}\}$ form a basis. □