

Math 2135 - Assignment 6

Due October 10, 2025

- (1) Prove the missing implication in the Invertible Matrix Theorem: If a square matrix A is invertible, then so is its transpose A^T .

Hint: What is the transpose of a product of two matrices?

Solution: Recall that $(AB)^T = B^T A^T$. Hence transposing

$$AA^{-1} = A^{-1}A = I_n$$

yields

$$(A^{-1})^T A^T = A^T (A^{-1})^T = I_n.$$

Hence $(A^{-1})^T$ is the inverse of A^T . □

- (2) Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.

Hint: Show that the columns of A are linearly dependent. Consider the cases $a = 0$ and $a \neq 0$ separately.

Solution: Assume $ad - bc = 0$.

Case, $a = 0$: Then $bc = 0$ yields $b = 0$ or $c = 0$. Hence

$$A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}.$$

Either way, the columns of A are linearly dependent.

Case, $a \neq 0$: Then $d = \frac{bc}{a}$. Hence

$$A = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$$

and the second column is $\frac{b}{a}$ times the first column. Hence the columns of A are linearly dependent.

By the Inverse Matrix Theorem, a matrix with linearly dependent columns is not invertible. □

- (3) Are the following invertible? Give the inverse if possible.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution:

$$A^{-1} = \frac{1}{2(-9) - 1 \cdot 4} \begin{bmatrix} -9 & -1 \\ -4 & 2 \end{bmatrix}, \quad B^{-1} \text{ does not exist since } 2(-6) - (-3)4 = 0$$

Since C has a zero column, for every matrix D the product DC has a zero column as well. So DC can never be the identity matrix. Thus C is not invertible. □

- (4) Compute the inverse if possible:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Solution: Since A is not square, it does not have an inverse.

Row reduce $[B, I_3]$:

$$\begin{aligned} \begin{bmatrix} -3 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -3 & 2 & 4 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -7 & 16 & 1 & 0 & 3 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & 0 & -2 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 10 & 4 \\ 0 & 1 & 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 1/2 & 7/2 & 3/2 \end{bmatrix} \end{aligned}$$

So

$$B^{-1} = \begin{bmatrix} 1 & 10 & 4 \\ 1 & 8 & 3 \\ 1/2 & 7/2 & 3/2 \end{bmatrix}.$$

For C^{-1} find the reduced echelon form of $[C, I_3]$:

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Since the echelon form of C has a zero row, C is not invertible. $\square$

- (5) Let A be an **upper triangular matrix**, that is,

$$A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{bmatrix}$$

with zeros below the diagonal. Show

- (a) A is invertible iff there are no zeros in the diagonal of A .
- (b) If A^{-1} exists, it is an upper triangular matrix as well.

Hint: When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, what happens to the n columns on the right?

Solution:

- (a) By the Invertible Matrix Theorem A is invertible iff the columns of A are linearly independent.

If the triangular matrix A has no zero diagonal entries, then A is actually in echelon form and its columns are linearly independent (hence A is invertible). Conversely if a diagonal entry of A is 0, then there is no pivot in this column of the echelon form of A . Hence the columns of A are not linearly independent (and A not invertible).

- (b) When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, we only need to obtain ones in the diagonal of A (by scaling rows) and zeros above the diagonal of A (by adding multiples of one row to rows above). These operations transform I_n into an upper triangular matrix A^{-1} . $\square$

- (6) Assume that $A \in \mathbb{R}^{n \times n}$ is invertible. Show that $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto A \cdot x$, is bijective.

Hint: Give a formula for the inverse function f^{-1} and check that it indeed describes the inverse of T .

Solution: $T^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto A^{-1} \cdot x$, is the inverse for T . This can be verified by composing $x \mapsto A^{-1}x$ with $T: x \mapsto Ax$ and observing that one gets the identity function on $\mathbb{R}^n$. $\square$

- (7) (a) What is the inverse of the rotation R by angle α counter clockwise around the origin in $\mathbb{R}^2$? What is the standard matrix of R^{-1} ?
 (b) What is the inverse of a reflection S on a line through the origin in $\mathbb{R}^2$? What can you say about the standard matrix B of S and its inverse? You do not have to write down B for this.

Solution:

- (a) R^{-1} is just the rotation by α clockwise (or by $-\alpha$ counter clockwise). R has standard matrix

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

and R^{-1} has standard matrix

$$A^{-1} = \frac{1}{\cos^2 \alpha + \sin^2 \alpha} \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}$$

- (b) Reflecting twice puts every point x back to itself. Hence any reflection is its own inverse, $S^{-1} = S$. the standard matrix B of S also satisfies $B^{-1} = B$. □

- (8) True or false? Explain your answer.

- (a) If A, B are square matrices with $AB = I_n$, then A and B are invertible.
 (b) Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent. Then $\mathbb{R}^n \rightarrow \mathbb{R}^n$, $x \mapsto Ax$ is not injective.

Solution:

- (a) True. By the Invertible Matrix Theorem $A^{-1} = B$ and $B^{-1} = A$.
 (b) True. If $Ax = b$ is inconsistent, then A does not have a pivot in every row. Since A is square, this means that it does not have a pivot in every column either. So $x \mapsto Ax$ is not injective. □