

Math 2135 - Assignment 5

Due October 2, 2026

- (1) If defined, compute the following for the matrices

$$A = \begin{bmatrix} 2 & 1 & -4 \\ 3 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Else explain why the computation is not defined.

- (a) AB (b) BA (c) AC (d) $A + C$ (e) $AB + 2C$

Solution:

$$AB = \begin{bmatrix} 2 \cdot 1 + 1 \cdot (-3) - 4 \cdot 2 & 2 \cdot (-1) + 1 \cdot 4 - 4 \cdot 0 \\ 3 \cdot 1 - 1 \cdot (-3) + 1 \cdot 2 & 3 \cdot (-1) - 1 \cdot 4 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -9 & 2 \\ 8 & -7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 \cdot 2 - 1 \cdot 3 & 1 \cdot 1 - 1 \cdot (-1) & 1 \cdot (-4) - 1 \cdot 1 \\ * & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} -1 & 2 & -5 \\ 6 & -7 & 16 \\ 4 & 2 & -8 \end{bmatrix}$$

AC is undefined since the length of A 's rows and the length of C 's columns are not the same.

$A + C$ is undefined since the sizes of A and C don't match.

$$AB + 2C = \begin{bmatrix} -9 & 2 \\ 8 & -7 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} -13 & 4 \\ 12 & -13 \end{bmatrix}$$

□

- (2) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ first rotate points around the origin by 60° counter clockwise and then reflect points at the line with equation $y = x$. Give the standard matrix for T .

- (a) Recall the standard matrix A for the rotation R by 60° from class.
(b) Determine the standard matrix B for the reflection S at the line with equation $y = x$ (a sketch will help).
(c) Since T is the composition of S and R , compute the standard matrix C of T as the product of B and A . Careful about the order!

Solution:

- (a) The standard matrix for the rotation R by $\alpha = 60^\circ$ is

$$A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

- (b) The reflection S at the diagonal flips the unit vectors, i.e., $T(e_1) = e_2$ and $T(e_2) = e_1$. Hence the standard matrix of S is

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- (c) Since $T = S \circ R$ (S after R), its standard matrix is

$$C = BA = \begin{bmatrix} \sin \alpha & \cos \alpha \\ \cos \alpha & -\sin \alpha \end{bmatrix}$$

□

- (3) Continuation of (2): What is the standard matrix for $U: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which first reflects points at the line with equation $y = x$ and then rotates points around the origin by 60° counter clockwise? Compare T and U .

Solution: Since $U = R \circ S$ (R after S), its standard matrix is

$$AB = \begin{bmatrix} -\sin \alpha & \cos \alpha \\ \cos \alpha & \sin \alpha \end{bmatrix}$$

Comparing the result with the standard matrix of $S \circ R$, we see that the result is not the same. Order of function composition and matrix multiplication matters! $\square$

- (4) Prove for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution: Multiplying $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and cancelling $ad - bc$ yields the identity matrix. Hence the given matrix is the inverse of A . $\square$

- (5) A **diagonal matrix** A has all entries 0 except on the diagonal, that is,

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Under which conditions is A invertible and what is A^{-1} ?

Solution: We see that

$$A^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{-1} \end{bmatrix}$$

is the only choice for the inverse of A , and it exists iff all diagonal entries $a_{11}, \dots, a_{nn}$ are distinct from 0. $\square$

- (6) Let $A, B \in \mathbb{R}^{n \times n}$ be invertible. Show $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

Solution: Multiplication yields $AB \cdot B^{-1}A^{-1} = A \cdot I_n \cdot A^{-1} = I_n$. Hence $B^{-1} \cdot A^{-1}$ is the inverse of AB . $\square$

- (7) A matrix $C \in \mathbb{R}^{n \times m}$ is called a **left inverse** of a matrix $A \in \mathbb{R}^{m \times n}$ if $CA = I_n$ (the $n \times n$ identity matrix).

- Show that if A has a left inverse C , then $Ax = b$ has at most one solution for any $b \in \mathbb{R}^m$.
- Give an example of a matrix A that has a left inverse but is not invertible and a vector b such that $Ax = b$ has no solution.

Solution:

- Multiply $Ax = b$ by C on the left to get $Cb = CAx = I_n x = x$. Hence $x = Cb$ is the only possible solution of $Ax = b$.

(b) You need a non-square matrix A for this. E.g. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has a left inverse $C = [1 \ 0]$ since $CA = [1]$. Still A is not invertible because there is no right inverse B such that $AB = I_2$ (alternatively because A is not square).

Also for $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution. Multiplying $Ax = b$ on the left by C yields $x = Cb = [0]$. So if $Ax = b$ had any solution at all, it could only be $x = [0]$. However this is clearly not a solution to the original system.

□