

Math 2135 - Assignment 4

Due September 25, 2026

- (1) Give the standard matrices for the following linear transformations:

(a) $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + y \\ x \\ -x + y \end{bmatrix};$

Solution: Just take the coefficient matrix of the transformation to get its standard matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$$

□

- (b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.

Solution: The function is $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}$ and has standard

matrix $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$

□

- (2) Is the following injective, surjective, bijective? What is its range?

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^2, x \mapsto \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot x$$

Solution: Not injective because x_1 is free in $A \cdot x = \mathbf{0}$. Alternatively, the columns of A are linearly dependent. So T is not injective (Theorem 12 of Section 1.9).

Surjective because A is in row echelon form and has no 0-rows (Theorem 12 of Section 1.9). Hence its range is just its codomain $\mathbb{R}^2$.

Bijective means injective and surjective. Hence T is not bijective because it is not injective. □

- (3) Is the following injective, surjective, bijective? What is its range?

$$T : \mathbb{R}^3 \rightarrow \mathbb{R}^3, x \mapsto \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot x$$

Solution: Row reduce the standard matrix of T to get

$$\begin{array}{ccc|ccc} 1 & -1 & 2 & & & \\ -2 & 0 & 1 & & & \\ 3 & -1 & 1 & & & \\ \hline 1 & -1 & 2 & & & \\ 0 & -2 & 5 & 2I + II & & \\ 0 & 2 & -5 & -3I + III & & \\ \hline 1 & -1 & 2 & & & \\ 0 & -2 & 5 & & & \\ 0 & 0 & 0 & -II + III & & \end{array}$$

T is not injective because not every column of the echelon form of A has a pivot. In particular x_3 is free in $A \cdot x = \mathbf{0}$.

T is not surjective because the echelon form of A has a zero row. Hence $Ax = y$ is not consistent for every $y \in \mathbb{R}^3$.

Since T is neither injective nor surjective, it is certainly not bijective.

The range of T is the span of the columns of A . Since the columns are linearly dependent it cannot be all of $\mathbb{R}^3$. Since the columns don't lie all on a line, they have to span a plane in $\mathbb{R}^3$, say spanned by the first two columns of A . $\square$

- (4) True or False? Explain why and correct the false statements to make them true.
- (a) A linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.
 - (b) Not every linear transformation $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $T(x) = Ax$ for some matrix A .
 - (c) The composition of any two linear transformations is linear as well.

Solution:

- (a) True because every vector is a linear combination of unit vectors.
- (b) False. Every linear $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ can be written as $T(x) = Ax$ for its standard matrix A .
- (c) True. If S, T are linear, then

$$S(T(u + v)) = S(T(u) + T(v)) = S(T(u) + S(T(v)))$$

for all u, v in the domain of T and

$$S(T(cu)) = S(cT(u)) = cS(T(u))$$

for $c \in \mathbb{R}$. Hence S composed with T is linear. $\square$

- (5) True or False? Explain why and correct the false statements to make them true.
- (a) $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $x \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.
 - (b) $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if every vector $x \in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.
 - (c) A linear map $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ cannot be one-to-one.
 - (d) There is a surjective linear transformation $T: \mathbb{R}^3 \rightarrow \mathbb{R}^4$.

Solution:

- (a) False. Any function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ maps every vector $x \in \mathbb{R}^n$ onto some vector $T(x)$ in $\mathbb{R}^m$.
The correct statement is: $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto $\mathbb{R}^m$ if for every vector $y \in \mathbb{R}^m$ there is some vector $x \in \mathbb{R}^n$ such that $T(x) = y$.
- (b) False. Any function $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ maps every vector $x \in \mathbb{R}^n$ onto the unique vector $T(x)$ in $\mathbb{R}^m$.
The correct statement is: $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if any 2 distinct vectors $x_1, x_2 \in \mathbb{R}^n$ are mapped to distinct vectors $T(x_1), T(x_2)$.
- (c) True. If A is the 2×3 standard matrix of T , then solving $A \cdot x = \mathbf{0}$ will always yield at least one free variable.
- (d) False. If A is the 4×3 standard matrix of T , then $Ax = y$ cannot have a solution for every $y \in \mathbb{R}^4$ since the echelon form of A has at least one zero row. $\square$

- (6) Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n, x \mapsto Ax$, be a surjective linear map. Show that T is injective as well.

Hint: Consider the pivots in A .

Solution: By a Theorem of class, if T is surjective, then A must have a pivot in every row. Since A is square, it then also has a pivot in every column. But that

means that the columns of A are linearly independent and that T is injective by the same Theorem. $\square$