Math 2135 - Practice Midterm 1

Except for problem 1, give full justifications and computations for all your answers!

(1) True or false?
- (a) If $Ax = b$ is inconsistent for some vector b, then A cannot have a pivot in every column.
- (b) If vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent and $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear independent.
- (c) The range of $T\colon \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$, is the span of the columns of A.
- (d) If the first two columns of a matrix B are equal, then so are the first two columns of AB.
- (e) There exist square matrices A, B such that neither A nor B is 0 (the matrix with all entries 0) but $AB = 0$.
- (f) $f\colon \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ 2y \end{bmatrix}$ is linear
- (g) The vectors $[1]$ and $[0]$ span $\mathbb{R}^1$.
- (h) If it is possible to reduce an augmented matrix $[A\ \mathbf{b}]$ to row echelon form, then $A\mathbf{x} = \mathbf{b}$ has at least one solution.

Solution:
- (a) False. If $Ax = b$ is inconsistent for some b, then the echelon form of A must have a zero row. So A cannot have a pivot in the last row. But it can still have pivots in every column if there are more rows than columns.
- (b) True. By a Theorem from class, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent iff one of the vectors is in the span of the previous vectors. Now assume $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. Then $\mathbf{v}_1$ is not 0 and $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$. Further assume $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$. Then no $\mathbf{v}_i$ is in the span of $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.
- (c) True. Ax is a linear combination of the columns of A, and $T(\mathbb{R}^n)$ is just the set of all these linear combinations, i.e., the span.
- (d) If $\mathbf{b}_i$ is the i-th column of B, then $A\mathbf{b}_i$ is the i-th column of AB. So if the first two columns of a matrix B are equal, then the first two columns of AB are equal as well.
- (e) E.g. $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Note AB is 0 but BA is not.
- (f) False. E.g. $f(0 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ but $0 \cdot f(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence f does not preserve scaling and is not linear.
- (g) True. Every $x \in \mathbb{R}$ can be written as $x = x \cdot 1 + 0 \cdot 0$.

(h) False. Every matrix can be reduced to echelon form but that does not mean the linear system has a solution.

□

(2) Let

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 4 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 5 \\ -8 \end{bmatrix}$$

(a) Give the solution for $Ax = b$ in parametrized vector form.
(b) Give vectors that span the null space of A.

Solution: (a) Row reduce the augmented matrix

$$[A\, b] = \begin{bmatrix} 0 & 3 & 1 & 2 & 6 \\ 1 & 4 & 0 & 7 & 5 \\ 2 & -1 & -3 & 8 & -8 \end{bmatrix} \text{ (flip rows 1 and 2 to eliminate in first column)}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 7 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 2 & -1 & -3 & 8 & -8 \end{bmatrix} \text{ (add (-2)*row 1 to row 3)}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 7 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 0 & -9 & -3 & -6 & -18 \end{bmatrix} \text{ (add 3*row 2 to row 3)}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 7 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $x_4 = t$ and $x_3 = s$ for $s, t \in \mathbb{R}$ are free. Next

$$3x_2 + s + 2t = 6 \text{ yields } x_2 = 2 - \frac{1}{3}s - \frac{2}{3}t$$

Finally

$$x_1 + 4(2 - \frac{1}{3}s - \frac{2}{3}t) + 0s + 7t = 5 \text{ yields } x_1 = -3 + \frac{4}{3}s - \frac{13}{3}t$$

Separating the solution into the constant part, multiples of s and of t yields the parametrized vector form

$$x = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -13/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$

(b) Note that $p = (-3, 2, 0, 0)^T$ above is a particular solution of $Ax = b$ and that

$$s \begin{bmatrix} 4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -13/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$

is the set of solutions of $Ax = 0$, i.e. the null space of A. Hence Nul $A = \text{Span}\{(4/3, -1/3, 1, 0)^T,\ (-13/3, -2/3, 0, 1)^T\}$. □

(3) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$, be an injective linear map. Show that T is surjective as well.

Solution: By a Theorem of class, if T is surjective, then A must have a pivot in every row. Since A is square, it then also has a pivot in every column. But that means that the columns of A are linearly independent and that T is injective by the same Theorem. □

(4) Is $\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$ in the span of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$?

Solution: No, the system

$$x \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

has no solution. □

(5) Give the standard matrix for the reflection T of $\mathbb{R}^2$ on the line $2x + 3y = 0$.

Solution: Note that $T(\begin{bmatrix} 3 \\ -2 \end{bmatrix}) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ since $b_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is on the line on which we reflect.

Further $T(\begin{bmatrix} 2 \\ 3 \end{bmatrix}) = -\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ since $b_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is perpendicular to the line.

From this we can compute the images of the unit vectors:

$$x \cdot b_1 + y \cdot b_2 = e_1$$

has solution $x = 3/13$ and $y = 2/13$. Hence by linearity

$$T(e_1) = \frac{3}{13} T(b_1) + \frac{2}{13} T(b_2) = \frac{1}{13} \begin{bmatrix} 5 \\ -12 \end{bmatrix}.$$

Similarly

$$-2/13 \cdot b_1 + 3/13 \cdot b_2 = e_2$$

yields

$$T(e_2) = -\frac{2}{13} T(b_1) + \frac{3}{13} T(b_2) = \frac{1}{13} \begin{bmatrix} -12 \\ -5 \end{bmatrix}.$$

The standard matrix A of T has $T(e_1)$ and $T(e_2)$ as columns. Hence

$$A = \frac{1}{13} \begin{bmatrix} 5 & -12 \\ -12 & -5 \end{bmatrix}.$$

□