

# Math 2135 - Assignment 12

Due Nov 21, 2024

(1) Are the matrices  $A, B, C, D$  in (4), (5), (6) of assignment 11 diagonalizable? How?

**Solution:**

$A$  is not diagonalizable because its eigenvalue  $-3$  has multiplicity 2 but the corresponding eigenspace only dimension 1.

$B$  is diagonalizable because it has 3 distinct eigenvalues, so

$$B = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$C$  is diagonalizable because it has 2 distinct eigenvalues, so

$$C = P \begin{bmatrix} 1 + \sqrt{6} & 0 \\ 0 & 1 - \sqrt{6} \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} 2 & -2 \\ \sqrt{6} & \sqrt{6} \end{bmatrix}$$

$D$  is not diagonalizable because its eigenvalue  $-3$  has multiplicity 2 but the corresponding eigenspace only dimension 1.  $\square$

(2) Let  $A$  be an  $n \times n$ -matrix. Are the following true or false? Explain why:

- (a) If  $A$  has  $n$  eigenvectors, then  $A$  is diagonalizable.
- (b) If a  $4 \times 4$ -matrix  $A$  has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then  $A$  is diagonalizable.
- (c)  $A$  is diagonalizable iff  $A$  has  $n$  eigenvalues (counting multiplicities).
- (d) If  $\mathbb{R}^n$  has a basis of eigenvectors of  $A$ , then  $A$  is diagonalizable.
- (e) Every triangular matrix is diagonalizable.

**Solution:**

- (a) **False.** You need  $n$  linearly independent eigenvectors.
- (b) **True.**
- (c) **False.** See example  $A$  in the previous problem.
- (d) **True.** A basis of  $\mathbb{R}^n$  of eigenvectors consists of  $n$  linearly independent eigenvectors.
- (e) **False.** See example  $A$  in the previous problem.  $\square$

(3) Let  $A$  be the standard matrix for the reflection  $t$  of  $\mathbb{R}^2$  on some line  $g$  through the origin. What are the eigenvalues, eigenvectors and eigenspaces of  $A$ ? Can  $A$  be diagonalized?

Hint: Consider what a reflection does to specific vectors.

**Solution:**

Let  $v_1$  be a non-zero vector on the line  $g$ , that is,  $v_1$  spans  $g$ . Then  $t(v_1) = Av_1 = v_1$ . Hence  $v_1$  is an eigenvector for  $A$  (equivalently for  $t$ ) with eigenvalue 1.

Let  $v_w$  be a non-zero vector orthogonal to  $g$ . Then  $t(v_w) = Av_w = -v_w$ . Hence  $v_w$  is an eigenvector for  $A$  (equivalently for  $t$ ) with eigenvalue  $-1$ .

Since  $A$  is a  $2 \times 2$ -matrix and has at most 2 eigenvalues we found all of them. Since  $v_1$  and  $v_w$  are non-zero and orthogonal, they form a basis  $B = (v_1, v_w)$  of  $\mathbb{R}^2$ . For  $P$

the matrix with columns  $v_1, v_2$ , we then have

$$A = P \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot P^{-1}.$$

Note that  $P$  is the change of coordinates matrix  $[id]_{B,E}$  and  $[t]_{B \leftarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ . So that's exactly how we computed  $[t]_{E \leftarrow E} = A$  earlier.  $\square$

(4) As the previous problem for a rotation  $r$  of  $\mathbb{R}^2$  by an angle  $\varphi$  around the origin.

Hint: Consider  $\varphi = 0, \pi$  separately.

**Solution:**

For  $\varphi = 0$  the rotation is just the identity map. The standard matrix is the identity matrix and already diagonalized. So every non-zero vector in  $\mathbb{R}^2$  is an eigenvector for eigenvalue 1, eigenspace  $E_1 = \mathbb{R}^2$ .

For  $\varphi = \pi$  the rotation is just scaling every vector by  $-1$ . The standard matrix is the negative of the identity matrix and already diagonalized. So every non-zero vector in  $\mathbb{R}^2$  is an eigenvector for eigenvalue  $-1$ , eigenspace  $E_{-1} = \mathbb{R}^2$ .

Else rotating a non-zero vector  $x$  by  $\varphi \neq 0, \pi$  does not give a real scalar multiple of  $x$ . Hence  $r$  has no real eigenvalues and no real eigenvectors;  $r$  cannot be diagonalized over the reals. Note that the characteristic polynomial of the standard matrix

$$\begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}$$

is  $\lambda^2 - 2 \cos \varphi \lambda + 1$  and has complex roots  $\cos \varphi \pm i \sin \varphi$ . It follows that  $r$  can be diagonalized over the complex numbers.  $\square$

(5) Consider a population of owls feeding on a population of squirrels. In month  $k$ , let  $o_k$  denote the number of owls and  $s_k$  the number of squirrels. Assume that the populations change every month as follows:

$$o_{k+1} = 0.3o_k + 0.4s_k$$

$$s_{k+1} = -0.4o_k + 1.3s_k$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let  $x_k = \begin{bmatrix} o_k \\ s_k \end{bmatrix}$ . Express the population change from  $x_k$  to  $x_{k+1}$  using a matrix  $A$ .

Diagonalize  $A$ .

**Solution:**

$$x_{k+1} = \underbrace{\begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 1.3 \end{bmatrix}}_A x_k$$

We diagonalize  $A$ . The characteristic equation is

$$0 = \det(A - \lambda I) = (0.3 - \lambda)(1.3 - \lambda) + 0.4^2 = \lambda^2 - 1.6\lambda + 0.55,$$

the eigenvalues are  $\lambda = \frac{1}{2}(1.6 \pm \sqrt{1.6^2 - 4 \cdot 0.55}) = 0.8 \pm 0.3 \in \{0.5, 1.1\}$ . We compute a basis for each eigenspace.

$$\lambda = 0.5 : \quad \text{Nul}(A - 0.5I) = \text{Nul} \begin{bmatrix} -0.2 & 0.4 \\ -0.4 & 0.8 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}.$$

$$\lambda = 1.1 : \quad \text{Nul}(A - 1.1I) = \text{Nul} \begin{bmatrix} -0.8 & 0.4 \\ -0.4 & 0.2 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}.$$

We write the eigenvectors in a matrix  $P$  and compute  $P^{-1}$ :

$$P = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}, \quad P^{-1} = \frac{1}{3/2} \begin{bmatrix} 1 & -1/2 \\ -1 & 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

We obtain a diagonalization

$$A = \underbrace{\begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}}_P \underbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & 1.1 \end{bmatrix}}_D \underbrace{\frac{1}{3} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}}_{P^{-1}}.$$

□

(6) Continue the previous problem: Let the starting population be  $x_1 = \begin{bmatrix} o_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$ .

- (a) Give an explicit formula for the populations in month  $k + 1$ .
- (b) Are the populations growing or decreasing over time? By which factor?
- (c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

**Solution:**

(a) (2 points)

$$\begin{aligned} x_{k+1} &= A^k x_1 = P D^k P^{-1} x_1 = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5^k & 0 \\ 0 & 1.1^k \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 20 \\ 100 \end{bmatrix} \\ &= \begin{bmatrix} 60 \cdot 1.1^k - 40 \cdot 0.5^k \\ 120 \cdot 1.1^k - 20 \cdot 0.5^k \end{bmatrix} \end{aligned}$$

(b) (2 points) Both populations are growing. For large  $k$ , the term  $0.5^k$  can be neglected (e.g. for  $k \geq 12$  we have  $1.1^k \geq 3.138$  and  $0.5^k \leq 0.00025$ ). We can approximate the populations by

$$x_{k+1} \approx \begin{bmatrix} 60 \cdot 1.1^k \\ 120 \cdot 1.1^k \end{bmatrix} = 1.1^k \begin{bmatrix} 60 \\ 120 \end{bmatrix} \quad \text{for large } k.$$

After a large number of months, both populations grow by a factor of 1.1 every month.

(c) (1 point) The populations are  $x_{13} = \begin{bmatrix} 188.3 \\ 376.6 \end{bmatrix}$  after 12 months and  $x_{25} = \begin{bmatrix} 591.0 \\ 1182.0 \end{bmatrix}$  after 24 months. After a large number of months, the ratio of owls to squirrels is always about 1 : 2 by the approximation formula for  $x_{k+1}$ .

□