Math 2135 - Assignment 11

Due November 14, 2025

You can use some computer algebra system to check your solutions for this assignment but have to show your calculations of characteristic polynomials, eigenvalues, etc, by hand.

(1) Let $A \in \mathbb{R}^{n\times n}$. Are the following true or false? Explain why:
  (a) If two rows or columns of A are identical, then $\det A = 0$.
  (b) For $c \in \mathbb{R}$, $\det(cA) = c \det A$.
  (c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.
  (d) A is invertible iff 0 is not an eigenvalue of A.

Solution:
(a) True. If two rows or columns of A are identical, then A is not invertible and $\det A = 0$.
(b) False. $\det(cA) = c^n \det A$ since in cA every row is multiplied by c.
(c) True. Assume A is invertible. Then $\det A \cdot A^{-1} = \det A \cdot \det A^{-1}$ by a Theorem from class. Since $\det A \cdot A^{-1} = \det I = 1$, the statement follows.
(d) True. By the Invertible Matrix Theorem A is invertible iff $\operatorname{Nul} A$ is trivial. The latter means that $\operatorname{Nul}(A - 0I) = \{0\}$, i.e. 0 is not an eigenvalue of A. □

(2) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h: V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

Solution:
We have to show that $E_{h,\lambda}$ contains the 0-vector, is closed under addition and scalar multiples. Using the linearity of h we get:
$0 \in E_{h,\lambda}$ since $h(0) = 0 = \lambda 0$
If $u, v \in E_{h,\lambda}$, then $h(u + v) = h(u) + h(v) = \lambda u + \lambda v = \lambda(u + v)$ and $u + v \in E_{h,\lambda}$.
If $v \in E_{h,\lambda}$ and $c \in F$, then $h(cv) = ch(v) = c\lambda v = \lambda cv$ and $cv \in E_{h,\lambda}$. □

(3) Are the following eigenvalues for the respective matrices? If so, give a basis for the corresponding eigenspace.

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}, \lambda = -5 \qquad B = \begin{bmatrix} 3 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \mu = 2$$

Solution:

(a) $A - (-5)I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$ has row echelon form $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and hence null space with basis $(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix})$. Since this nullspace is non-trivial, it is the eigenspace for eigenvalue -5 of A.

(b) $B - 2I = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -4 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is invertible. Hence it has trivial nullspace and 2 is not an eigenvalue for B.

□

(4) Give all eigenvalues and bases for eigenspaces of the following matrices. Do you need the characteristic polynomials?

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

Solution:

Since A, B are triangular matrices, their eigenvalues are just their diagonal elements.

(a) A has eigenvalue -3 with multiplicity 2: $\text{Nul}(A - (-3)I)$ has basis $(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

(b) B has eigenvalues $2, 0, 3$:
$\text{Nul}(A - 2I)$ has basis $(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix})$.
$\text{Nul}(A - 0I)$ has basis $(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$.
$\text{Nul}(A - 3I)$ has basis $(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$.

□

(5) Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

Solution:

Characteristic polynomial:

$$\begin{aligned} \det(C - \lambda I) &= (1-\lambda)(1-\lambda) - 2 \cdot 3 \\ &= \lambda^2 - 2\lambda - 5 \end{aligned}$$

Eigenvalues are the roots of the characteristic polynomial. Use the quadratic formula

$$\begin{aligned} \lambda_{1,2} &= 1 \pm \sqrt{1+5} \\ &= 1 \pm \sqrt{6} \end{aligned}$$

Eigenvector for $\lambda = 1 + \sqrt{6}$:

$$C - \lambda I = \begin{bmatrix} -\sqrt{6} & 2 \\ 3 & -\sqrt{6} \end{bmatrix} \sim \begin{bmatrix} -\sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and added to row 2.

So the eigenspace for $\lambda = 1 + \sqrt{6}$ has basis $(\begin{bmatrix} 2 \\ \sqrt{6} \end{bmatrix})$.

Eigenvector for $\lambda = 1 - \sqrt{6}$:

$$C - \lambda I = \begin{bmatrix} \sqrt{6} & 2 \\ 3 & \sqrt{6} \end{bmatrix} \sim \begin{bmatrix} \sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and subtracted from row 2.

So the eigenspace for $\lambda = 1 - \sqrt{6}$ has basis $(\begin{bmatrix} -2 \\ \sqrt{6} \end{bmatrix})$.

□

(6) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

Solution:
Characteristic polynomial:

$$\begin{aligned} \det(D - \lambda I) &= (-3 - \lambda) \cdot \det \begin{bmatrix} -1 - \lambda & 4 \\ 6 & 9 - \lambda \end{bmatrix} \\ &(-3 - \lambda)[(-1 - \lambda)(9 - \lambda) - 24] \\ &(-3 - \lambda)[\lambda^2 - 8\lambda - 33] \end{aligned}$$

Eigenvalues are $\lambda_1 = -3$ and the roots of $\lambda^2 - 8\lambda - 33$. The quadratic formula yields

$$\lambda_{2,3} = 4 \pm \sqrt{4^2 + 33}$$

So $\lambda_2 = -3$ and $\lambda_3 = 11$.

The eigenspace for $\lambda = -3$ has basis $(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix})$.

The eigenspace for $\lambda = 11$ has basis $(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix})$. □

(7) Let $A \in \mathbb{R}^{n \times n}$ with n eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to their multiplicities). Show that

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

Hint: Note that the characteristic polynomial of A can be factored as

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

Why? Check that the signs are correct.

Solution:
Since $\lambda_1, \ldots, \lambda_n$ are the roots of the characteristic polynomial, the characteristic polynomial can be factored as

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

Note that the signs are correct because on both sides of the equation the coefficient of λ^n is $(-1)^n$.

By plugging in $\lambda = 0$ we get

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

□