Math 2135 - Assignment 6

Due October 10, 2025

(1) Prove the missing implication in the Invertible Matrix Theorem: If a square matrix A is invertible, then so is it transpose A^T.
Hint: What is the transpose of a product of two matrices?

Solution: Recall that $(AB)^T = B^T A^T$. Hence transposing

$$AA^{-1} = A^{-1}A = I_n$$

yields

$$(A^{-1})^T A^T = A^T (A^{-1})T = I_n.$$

Hence $(A^{-1})^T$ is the inverse of A^T. □

(2) Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.
Hint: Show that the columns of A are linearly dependent. Consider the cases $a = 0$ and $a \neq 0$ separately.

Solution: Assume $ad - bc = 0$.
Case, $a = 0$: Then $bc = 0$ yields $b = 0$ or $c = 0$. Hence

$$A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}.$$

Either way, the columns of A are linearly dependent.
Case, $a \neq 0$: Then $d = \frac{bc}{a}$. Hence

$$A = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$$

and the second column is $\frac{b}{a}$ times the first column. Hence the columns of A are linearly dependent.

By the Inverse Matrix Theorem, a matrix with linearly dependent columns is not invertible. □

(3) Let A be an **upper triangular matrix**, that is,

$$A = \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

with zeros below the diagonal. Show
(a) A is invertible iff there are no zeros in the diagonal of A.
(b) If A^{-1} exists, it is an upper triangular matrix as well.
Hint: When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, what happens to the n columns on the right?

Solution:
(a) By the Invertible Matrix Theorem A is invertible iff the columns of A are linearly independent.
If the triangular matrix A has no zero diagonal entries, then A is actually in echelon form and its columns are linearly independent (hence A is invertible).

Conversely if a diagonal entry of A is 0, then there is no pivot in this column of the echelon form of A. Hence the columns of A are not linearly independent (and A not invertible).

(b) When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, we only need to obtain ones in the diagonal of A (by scaling rows) and zeros above the diagonal of A (by adding multiples of one row to rows above). These operations transform I_n into an upper triangular matrix A^{-1}.

□

(4) Assume that $A \in \mathbb{R}^{n\times n}$ is invertible. Show that $T\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto A \cdot x$, is bijective.

Hint: Give a formula for the inverse function f^{-1} and check that it indeed describes the inverse of T.

Solution: $T^{-1}\colon F^n \to F^n,\ x \mapsto A^{-1} \cdot x$, is the inverse for T. This can be verified by composing $x \to A^{-1}x$ with $T\colon x \to Ax$ and observing that one gets the identity function on $\mathbb{R}^n$. □

(5) (a) What is the inverse of the rotation R by angle α counter clockwise around the origin in $\mathbb{R}^2$? What is the standard matrix of R^{-1}?

(b) What is the inverse of a reflection S on a line through the origin in $\mathbb{R}^2$? What can you say about the standard matrix B of S and its inverse? You do not have to write down B for this.

Solution:

(a) R^{-1} is just the rotation by α clockwise (or by $-\alpha$ counter clockwise). R has standard matrix

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

and R^{-1} has standard matrix

$$A^{-1} = \frac{1}{\cos\alpha^2 + \sin\alpha^2} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}$$

(b) Reflecting twice puts every point x back to itself. Hence any reflection is its own inverse, $S^{-1} = S$. the standard matrix B of S also satisfies $B^{-1} = B$.

□

(6) True of false? Explain your answer.

(a) If A, B are square matrices with $AB = I_n$, then A and B are invertible.

(b) Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent. Then $\mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto Ax$ is not injective.

Solution:

(a) True. By the Invertible Matrix Theorem $A^{-1} = B$ and $B^{-1} = A$.

(b) True. If $Ax = b$ is inconsistent, then A does not have a pivot in every row. Since A is square, this means that it does not have a pivot in every column either. So $x \mapsto Ax$ is not injective.

□

(7) Explain why the following are not subspaces of $\mathbb{R}^2$. Give explicit counter examples for subspace properties that are not satisfied.

(a) $U = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, x \geq 0\}$

(b) $V = \mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of all integers)

(c) $W = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, |x| = |y|\}$ ($|x|$ denotes the absolute value of x).

Solution:

(a) Not closed under scalar multiples, e.g. $\begin{bmatrix}1\\0\end{bmatrix} \in U$ but $(-1)\begin{bmatrix}1\\0\end{bmatrix} \notin U$

(b) Not closed under scalar multiples, e.g. $\begin{bmatrix}1\\0\end{bmatrix} \in V$ but $\sqrt{2}\begin{bmatrix}1\\0\end{bmatrix} \notin V$

(c) Not closed under addition, e.g. $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix} \in W$ but $\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix} \notin W$

□

(8) Let $A \in \mathbb{R}^{m\times n}$. Prove that $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$.

Solution: We show the 3 conditions for being a subspace.

(a) The zero vector is clearly in $\mathrm{Null}(A)$ since $A\mathbf{0} = \mathbf{0}$.

(b) Let $\mathbf{u}$ and w be arbitary vectors in $\mathrm{Null}(A)$. Then $A\mathbf{u} = \mathbf{0}$ and $Aw = \mathbf{0}$. We show that $\mathbf{u} + w$ is in $\mathrm{Null}(A)$.

$$A(\mathbf{u} + w) = A\mathbf{u} + Aw = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So $\mathbf{u} + w$ is in $\mathrm{Null}(A)$.

(c) Let $r \in \mathbb{R}$. Then

$$A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}.$$

Hence $r\mathbf{u}$ is in $\mathrm{Null}(A)$.

□