Math 2135 - Assignment 6

Due October 10, 2025

(1) Prove the missing implication in the Invertible Matrix Theorem: If a square matrix A is invertible, then so is it transpose A^T.
Hint: What is the transpose of a product of two matrices?

(2) Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.
Hint: Show that the columns of A are linearly dependent. Consider the cases $a = 0$ and $a \neq 0$ separately.

(3) Let A be an **upper triangular matrix**, that is,

$$A = \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

with zeros below the diagonal. Show
  (a) A is invertible iff there are no zeros in the diagonal of A.
  (b) If A^{-1} exists, it is an upper triangular matrix as well.
  Hint: When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, what happens to the n columns on the right?

(4) Assume that $A \in \mathbb{R}^{n \times n}$ is invertible. Show that $T\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto A \cdot x$, is bijective.
Hint: Give a formula for the inverse function f^{-1} and check that it indeed describes the inverse of T.

(5) (a) What is the inverse of the rotation R by angle α counter clockwise around the origin in $\mathbb{R}^2$? What is the standard matrix of R^{-1}?
  (b) What is the inverse of a reflection S on a line through the origin in $\mathbb{R}^2$? What can you say about the standard matrix B of S and its inverse? You do not have to write down B for this.

(6) True of false? Explain your answer.
  (a) If A, B are square matrices with $AB = I_n$, then A and B are invertible.
  (b) Let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent. Then $\mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto Ax$ is not injective.

(7) Explain why the following are not subspaces of $\mathbb{R}^2$. Give explicit counter examples for subspace properties that are not satisfied.
  (a) $U = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, x \geq 0 \}$
  (b) $V = \mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of all integers)
  (c) $W = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, |x| = |y| \}$ ($|x|$ denotes the absolute value of x).

(8) Let $A \in \mathbb{R}^{m \times n}$. Prove that $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$.