Math 2135 - Assignment 5

Due October 4, 2024

You can check your results using Mathematica but do the calculations by hand and show them to receive credit.

(1) Prove for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ that

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution: Multiplying $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and cancelling $ad - bc$ yields the identity matrix. Hence the given matrix is the inverse of A.

□

(2) Are the following invertible? Give the inverse if possible.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution:

$$A^{-1} = \frac{1}{2(-9) - 1 \cdot 4} \begin{bmatrix} -9 & -1 \\ -4 & 2 \end{bmatrix}, \quad B^{-1} \text{ does not exist since } 2(-6) - (-3)4 = 0$$

Since C has a zero column, for every matrix D the product DC has a zero column as well. So DC can never be the identity matrix. Thus C is not invertible.

□

(3) A **diagonal matrix** A has all entries 0 except on the diagonal, that is,

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Under which conditions is A invertible and what is A^{-1}?

Solution: We see that

$$A^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{-1} \end{bmatrix}$$

is the only choice for the inverse of A, and it exists iff all diagonal entries $a_{11}, \dots, a_{nn}$ are distinct from 0. □

(4) Compute the inverse if possible:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Solution: Since A is not square, it does not have an inverse.

Row reduce $[B, I_3]$:

$$\begin{bmatrix} -3 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -3 & 2 & 4 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -7 & 16 & 1 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 10 & 4 \\ 0 & 1 & 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 1/2 & 7/2 & 3/2 \end{bmatrix}$$

So

$$B^{-1} = \begin{bmatrix} 1 & 10 & 4 \\ 1 & 8 & 3 \\ 1/2 & 7/2 & 3/2 \end{bmatrix}.$$

For C^{-1} find the reduced echelon form of $[C, I_3]$:

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Since the echelon form of C has a zero row, C is not invertible. □

(5) Let $A, B \in \mathbb{R}^{n\times n}$ be invertible. Show $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

Solution: Multiplication yields $AB \cdot B^{-1}A^{-1} = A \cdot I_n \cdot A^{-1} = I_n$. Hence $B^{-1} \cdot A^{-1}$ is the inverse of AB. □

(6) A matrix $C \in \mathbb{R}^{n\times m}$ is called a **left inverse** of a matrix $A \in \mathbb{R}^{m\times n}$ if $CA = I_n$ (the $n \times n$ identity matrix).

(a) Show that if A has a left inverse C, then $Ax = b$ has at most one solution for any $b \in \mathbb{R}^m$.

(b) Give an example of a matrix A that has a left inverse but is not invertible and a vector b such that $Ax = b$ has no solution.

Solution:

(a) Multiply $Ax = b$ by C on the left to get $Cb = CAx = I_n x = x$. Hence $x = Cb$ is the only possible solution of $Ax = b$.

(b) You need a non-square matrix A for this. E.g. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has a left inverse $C = [1\ 0]$ since $CA = [1]$. Still A is not invertible because there is no right inverse B such that $AB = I_2$ (alternatively because A is not square).

Also for $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ has no solution. Multiplying $Ax = b$ on the left by C yields $x = Cb = [0]$. So if $Ax = b$ had any solution at all, it could only be $x = [0]$. However this is clearly not a solution to the original system.

□