Math 2135 - Assignment 4

Due September 26, 2025

(1) Is the following injective, surjective, bijective? What is its range (image)?

$$T : \mathbb{R}^3 \to \mathbb{R}^2,\ x \mapsto \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot x$$

(2) Is the following injective, surjective, bijective? What is its range (image)?

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot x$$

(3) True or False? Explain why and correct the false statements to make them true.
  (a) If vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent and $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear independent.
  (b) A linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.
  (c) Not every linear transformation $T \colon \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(x) = Ax$ for some matrix A.
  (d) The composition of any two linear transformations is linear as well.

(4) True or False? Explain why and correct the false statements to make them true.
  (a) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $x \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.
  (b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $x \in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.
  (c) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one.
  (d) There is a surjective linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^4$.

(5) Let $T \colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$, be a surjective linear map. Show that T is injective as well.
Hint: Consider the pivots in A.

(6) If defined, compute the following for the matrices

$$A = \begin{bmatrix} 2 & 1 & -4 \\ 3 & -1 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ 2 & 0 \end{bmatrix},\ C = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Else explain why the computation is not defined.
(a) AB (b) BA (c) AC (d) $A + C$ (e) $AB + 2C$

(7) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ first rotate points around the origin by $60°$ counter clockwise and then reflect points at the line with equation $y = x$. Give the standard matrix for T following these steps:
  (a) Recall the standard matrix A for the rotation R by $60°$ from class.
  (b) Determine the standard matrix B for the reflection S at the line with equation $y = x$ (a sketch will help).
  (c) Since T is the composition of S and R, compute the standard matrix C of T as the product of B and A. Careful about the order!

(8) Continuation of (7): What is the standard matrix for $U \colon \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points at the line with equation $y = x$ and then rotates points around the origin by $60°$ counter clockwise? Compare T and U.