

Math 2135 - Assignment 3

Due September 19, 2025

- (1) Which of the following sets of vectors are linearly independent?

(a) $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 0 \end{bmatrix}$

Solution:

Check whether $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a non-trivial solution:

- (a) Row reduce

$$\begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 7 & -2 \end{bmatrix} \sim \dots$$

Note this is the coefficient matrix of the linear system, not the augmented matrix but there the last column is all 0s anyway.

3 pivots, no free variables for the null space, columns are linearly independent.

- (b)

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & -11 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -14 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots, 1 free variable for the null space, columns are linearly dependent.

□

- (2) Explain whether the following are true or false:

- (a) Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3$.
- (b) A subset $\{\mathbf{v}\}$ containing just a single vector is linearly dependent iff $\mathbf{v} = \mathbf{0}$.
- (c) Two vectors are linearly dependent iff they lie on a line through the origin.
- (d) There exist four vectors in $\mathbb{R}^3$ that are linearly independent.

Solution:

- (a) **True.** If $\mathbf{v}_2 = c_1\mathbf{v}_1 + c_3\mathbf{v}_3$ for some $c_1, c_3 \in \mathbb{R}$, then $c_1\mathbf{v}_1 + (-1)\mathbf{v}_2 + c_3\mathbf{v}_3$ is a non-trivial linear combination that yields the zero vector $\mathbf{0}$.
- (b) **True.** Assume $\mathbf{v} = \mathbf{0}$ is the zero vector. Then $1\mathbf{v} = \mathbf{0}$ is a non-trivial linear combination that yields the zero vector. Hence $\mathbf{v}$ is linearly dependent. Conversely, assume $\mathbf{v} \neq \mathbf{0}$. Then $c\mathbf{v} = \mathbf{0}$ only if $c = 0$. Hence $\mathbf{v}$ is linearly independent.
- (c) **True.** Assume $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent. Then either $\mathbf{v}_1 = \mathbf{0}$ or $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$ by a Theorem from class. In either case $\mathbf{v}_1, \mathbf{v}_2$ lie in a line through the origin. Conversely, assume $\mathbf{v}_1, \mathbf{v}_2$ lie in a line through the origin. Then one is a multiple of the other and both are linearly dependent.
- (d) **False.** For $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{R}^3$ solving $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = \mathbf{0}$ yields a coefficient matrix with 3 rows and 4 columns. Since there are only 3 rows, there can be at most 3 pivots. Hence there is at least 1 free variable in the solution of the system. Hence $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are linearly dependent.

□

- (3) Show: If any of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the zero vector (say $\mathbf{v}_i = \mathbf{0}$ for $i \leq n$), then $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.

Solution:

Assume $\mathbf{v}_i = \mathbf{0}$. Then

$$0\mathbf{v}_1 + \dots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \dots + 0\mathbf{v}_n = \mathbf{0}$$

is a non-trivial linear combination that yields the zero vector. Hence $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent. □

- (4) Show: If $n > m$, then any n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$ are linearly dependent.

Solution:

Compare with (2)(d). Solving $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$ yields a coefficient matrix with m rows and n columns. There can be at most m pivots. Hence there are at least $n - m > 0$ free variables in the solution of the system. Hence $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly dependent. □

- (5) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.

$$\begin{aligned} \text{(a)} \quad f: \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+3 \end{bmatrix} \\ \text{(b)} \quad g: \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix} \\ \text{(c)} \quad h: \mathbb{R}^2 &\rightarrow \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix} \end{aligned}$$

Solution:

For example

$$\begin{aligned} \text{(a)} \quad f(0 \cdot \mathbf{0}) &= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \mathbf{0} = 0 \cdot f(\mathbf{0}) \\ \text{(b)} \quad g(2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot g\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ \text{(c)} \quad h((-1) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}) &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \neq \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1) \cdot h\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \end{aligned}$$

□

- (6) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear map such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Use the linearity of T to compute $T\left(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$. What is the issue with the latter?

Solution:

$$T\left(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}\right) = T\left(2 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = 2 \cdot T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + 3 \cdot T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = 2 \cdot \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -11 \\ 4 \\ 3 \end{bmatrix}$$

$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$ cannot be computed with the given information since $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. □

(7) Let $T: \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear map such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \quad T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

(a) Use the linearity of T to find $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

(b) Determine $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ for arbitrary $x, y \in \mathbb{R}$.

Solution:

(a) First write the unit vectors as linear combinations of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Solve

$$x \begin{bmatrix} 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to get $x = -\frac{1}{2}$ and $y = \frac{1}{2}$. By the linearity of T we obtain

$$\begin{aligned} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= T\left(-\frac{1}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) \\ &= -\frac{1}{2} T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) + \frac{1}{2} T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) \\ &= -\frac{1}{2} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

Similarly we compute that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and hence obtain

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \frac{3}{4} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1/2 \\ -5/2 \end{bmatrix}$$

(b) By (a) we know the standard matrix of T is

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1/2 \\ 2 & -5/2 \end{bmatrix}.$$

$$\text{Thus } T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}.$$

□

(8) Give the standard matrices for the following linear transformations:

$$(a) \ T : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + y \\ x \\ -x + y \end{bmatrix};$$

Solution:

Just take the coefficient matrix of the transformation to get its standard matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}$$

□

(b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.

Solution:

The function is $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}$ and has standard matrix $A = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

□