

## 2.2 Inverse matrices.

**Definition.** An  $n \times n$ -matrix  $A$  is *invertible* if there exists an  $n \times n$ -matrix  $B$  such that

$$AB = BA = I_n.$$

Then  $B$  is called the *inverse* of  $A$  and denoted by  $A^{-1}$ .

Note: If an inverse of  $A$  exists, it is unique.

**Theorem.** Let  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . If  $ad - bc \neq 0$ , then  $A$  is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Else if  $ad - bc = 0$ , then  $A$  is not invertible.

*Proof.* HW

**Example.**  $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$A^{-1} = \frac{1}{3 \cdot 6 - 4 \cdot 5} \begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

**Theorem.** For invertible matrices  $A, B$

$$(1) (A^{-1})^{-1} = A$$

$$(2) (AB)^{-1} = B^{-1} A^{-1}$$

*Proof.* 1) By  $A \cdot A^{-1} = A^{-1} \cdot A = I_n$  we have  $A^{-1}$  is invertible with inverse  $A$ .

2) HW

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## 2.2 Characterizations of invertible matrices.

**Question.** How to recognize that a matrix has an inverse?

**Invertible Matrix Theorem.** For an  $A \in \mathbb{R}^{n \times n}$  the following are equivalent:

- (1)  $A$  is invertible.
- (2)  $A$  can be row reduced to  $I_n$ .
- (3) The columns of  $A$  are linearly independent.
- (4) The columns of  $A$  span  $\mathbb{R}^n$ .
- (5) There exists a *left inverse*  $C \in \mathbb{R}^{n \times n}$  of  $A$  such that  $CA = I_n$ .
- (6) There exists a *right inverse*  $D \in \mathbb{R}^{n \times n}$  of  $A$  such that  $AD = I_n$ .
- (7) The transpose  $A^T$  is invertible.

*Proof.* 1)  $\Rightarrow$  5) : for  $C = A^{-1}$

5)  $\Rightarrow$  3) By HW  $Ax = 0$  implies  
 $\underbrace{CA}_{I_n}x = C0 = 0$ , hence  $x = 0$ .

Null  $A = \{0\}$  and the columns of  $A$  linearly independent.

3)  $\Rightarrow$  2) Since  $Ax = 0$  has no free variables, the echelon form of  $A$  is  $\begin{bmatrix} \times & & & \\ & \times & & \\ & & \times & \\ & & & \times \end{bmatrix}$  with a pivot in every column and every row. Using row operations this can be reduced to  $I_n$ .

2)  $\Rightarrow$  4) clear since the row echelon form of  $A$  has no zero rows.

4)  $\Rightarrow$  6) Find i-th column  $d_i$  of  $D$  as solution for  
 $A d_i = e_i$  for  $1 \leq i \leq n$ .  
 (consistent by assumption 4.)

Note we have proved 5)  $\Rightarrow$  6) so far.

5)  $\Rightarrow$  1) Let  $CA = I_n$ . By 5)  $\Rightarrow$  6) we have  $D$  such that  
 $AD = I_n$

$$\text{Then } C = \underbrace{C(A^{-1}D)}_{=I_n} = \underbrace{(CA)}_{I_n} D = D$$

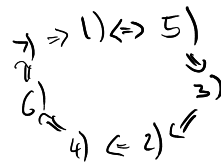
Hence  $CA = AC = I_n$  and  $A$  is invertible.

Note we have proved 1)  $\Leftrightarrow$  5) so far.

6)  $\Rightarrow$  7)  $AD = I_n$  yields  $(AD)^T = D^T A^T = I_n$ .  
 By 5)  $\Rightarrow$  1)  $A^T$  is invertible

7)  $\Rightarrow$  1) HW.

Structure of the proof:



3

□

**Example.** Is  $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$  invertible?

Row reduce  $A$

$$\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \\ \hline 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}$$

No zeros in echelon form  $\Rightarrow$  can be reduced to  $I_n$   
 no free variables, columns are linearly independent.  
 $A$  is invertible.

**Recall.**  $f: A \rightarrow B$  is bijective iff  $f$  has an *inverse function*  $f^{-1}: B \rightarrow A$  such that

$$\begin{aligned} f^{-1}(f(x)) &= x \text{ for all } x \in A, \\ f(f^{-1}(y)) &= y \text{ for all } y \in B. \end{aligned}$$

**Theorem.** A linear map  $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $x \mapsto Ax$ , is bijective (invertible) iff  $A$  is invertible.

(Then  $T^{-1}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $x \mapsto A^{-1}x$ , still linear).

*Proof.*  $\Rightarrow$  Assume  $T$  is bijective.

Then  $T$  is onto  $\mathbb{R}^n$ .

$Ax = b$  has a solution for every  $b \in \mathbb{R}^n$

By the Invertible Matrix Thm 4)  $\Rightarrow$  1),  $A$  is invertible.

$\Leftarrow$  HW.

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