2.2 Inverse matrices.

Definition. An $n\times n$-matrix A is *invertible* if there exists an $n\times n$-matrix B such that

$$AB = BA = I_n.$$

Then B is called the *inverse* of A and denoted by A^{-1}.

Note: If an inverse of A exists, it is unique.

Theorem. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

- swap elements in diagonal
- change signs off diagonal
- divide by the product of diagonal elements minus the product of off diagonal elements

Else if $ad - bc = 0$, *then* A *is not invertible.*

Proof. HW

Example. $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$A^{-1} = \frac{1}{3\cdot 6 - 4\cdot 5}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix}$$

Theorem. *For invertible matrices* A, B

(1) $(A^{-1})^{-1} = A$

(2) $(AB)^{-1} = B^{-1}A^{-1}$

Proof. 1) By $A\cdot A^{-1} = A^{-1}\cdot A = I_n$ we have A^{-1} is invertible with inverse A.

2) HW

2.2 Characterizations of invertible matrices.

Question. How to recognize that a matrix has an inverse?

Invertible Matrix Theorem. For an $A \in \mathbb{R}^{n\times n}$ the following are equivalent:

(1) A is invertible.
(2) A can be row reduced to I_n.
(3) The columns of A are linearly independent.
(4) The columns of A span $\mathbb{R}^n$.
(5) There exists a *left inverse* $C \in \mathbb{R}^{n\times n}$ of A such that $CA = I_n$.
(6) There exists a *right inverse* $D \in \mathbb{R}^{n\times n}$ of A such that $AD = I_n$.
(7) The transpose A^T is invertible.

Proof. 1) $\Rightarrow$ 5): for $C = A^{-1}$

5) $\Rightarrow$ 3) By HW $Ax = 0$ implies

$$\underbrace{CA}_{I_n}x = C0 = 0, \text{ hence } x = 0.$$

Null $A = \{0\}$ and the columns of A linearly independent.

3) $\Rightarrow$ 2) Since $Ax = 0$ has no free variables, the echelon form of A is $\begin{bmatrix} \blacksquare & * & & & \\ & \blacksquare & * & & \\ & & \blacksquare & * & \\ & & & \blacksquare & * \\ & & & & \blacksquare \end{bmatrix}$ with a pivot in every column and every row. Using row operations this can be reduced to I_n.

2) $\Rightarrow$ 4) clear since the row echelon form of A has no zero rows.

4) $\Rightarrow$ 6) Find i-th column d_i of D as solution for

$$A d_i = e_i \quad \text{for } 1 \le i \le n.$$

(consistent by assumption 4.)

Note we have proved 5) $\Rightarrow$ 6) so far.

5) $\Rightarrow$ 1) Let $CA = I_n$. By 5) $\Rightarrow$ 6) we have D such that

$$AD = I_n$$

Then $$C = C\underbrace{(AD)}_{=I_n} = \underbrace{(CA)}_{I_n}D = D$$

Hence $CA = AC = I_n$ and A is invertible.

Note we have proved 1) $\Leftrightarrow$ 5) so far.

6) $\Rightarrow$ 7) $AD = I_n$ yields $(AD)^T = D^T A^T = I_n$.

By 5) $\Rightarrow$ 1) A^T is invertible

7) $\Rightarrow$ 1) HW.

Structure of the proof:

$$\begin{array}{ccccc} 7) & \Rightarrow & 1) & \Leftrightarrow & 5) \\ \Uparrow & & & & \Downarrow \\ 6) & & & & 3) \\ \Uparrow & & & & \Downarrow \\ 4) & & \Leftarrow & & 2) \end{array}$$

$\square$

Example. Is $A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix}$ invertible?

Row reduce A

$$\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & -1 & -1 \\ \hline 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{array}$$

No zero rows in echelon form $\Rightarrow$ can be reduced to I_n
no free variables, columns are lin independent.
A is invertible.

Recall. $f\colon A \to B$ is bijective iff f has an *inverse function* $f^{-1}\colon B \to A$ such that

$$f^{-1}\big(f(x)\big) = x \text{ for all } x \in A,$$
$$f\big(f^{-1}(y)\big) = y \text{ for all } y \in B.$$

Theorem. *A linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$, is bijective (invertible) iff A is invertible.*
(Then $T^{-1}\colon \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto A^{-1}x$, still linear).

Proof. $\Rightarrow$ Assume T is bijective.
Then T is onto $\mathbb{R}^n$.
$Ax = b$ has a solution for every $b \in \mathbb{R}^n$
By the Invertible Matrix Thm 4) $\Rightarrow$ 1), A is invertible.

$\Leftarrow$ HW. $\square$