Math 2135 - Practice Final

December, 2024

(1) Let $B = (\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix})$.

(a) Why is B a basis of $\mathbb{R}^2$?

(b) Give change of coordinates matrices $P_{E\leftarrow B}$ (for changing B-coordinates into coordinates w.r.t. the standard basis E) and $P_{B\leftarrow E}$.

(c) Compute the coordinates $[x]_B$ for $x = \begin{bmatrix}2\\3\end{bmatrix}$.

Solution:

(a) B is a basis since it contains 2 linear independent vectors of $\mathbb{R}^2$

(b)

$$P_{E\leftarrow B} = [b_1, b_2] = \begin{bmatrix}1 & 3\\2 & 4\end{bmatrix} \text{ and } P_{B\leftarrow E} = [b_1, b_2]^{-1} = \frac{1}{4-6}\begin{bmatrix}4 & -3\\-2 & 1\end{bmatrix}$$

(c)

$$[x]_B = P_{B\leftarrow E}\cdot x = \frac{1}{2}\begin{bmatrix}-4 & 3\\2 & -1\end{bmatrix}\cdot\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1/2\\1/2\end{bmatrix}$$

□

(2) Let $B = (b_1, b_2)$ as in the previous problem. Let $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ be linear such that $[h(b_1)]_E = \begin{bmatrix}-1\\1\end{bmatrix}, [h(b_2)]_E = \begin{bmatrix}0\\1\end{bmatrix}$.

(a) Give the standard matrix $T_{E\leftarrow E}$ of h w.r.t. the standard basis.

(b) Compute $h(\begin{bmatrix}1\\1\end{bmatrix})$.

Solution:

(a) Recall that the matrix of h w.r.t B and E is

$$T_{E\leftarrow B} = [[h(b_1)]_E\ [h(b_2)]_E] = \begin{bmatrix}-1 & 0\\1 & 1\end{bmatrix}$$

We only have to change the first basis from B to E via the change of basis matrix $T_{B\leftarrow E}$,

$$T_{E\leftarrow E} = T_{E\leftarrow B}\cdot P_{B\leftarrow E} = \begin{bmatrix}-1 & 0\\1 & 1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}-4 & 3\\2 & -1\end{bmatrix} = \begin{bmatrix}2 & -3/2\\-1 & 1\end{bmatrix}$$

(b)

$$h(\begin{bmatrix}1\\1\end{bmatrix}) = T_{E\leftarrow E}\cdot\begin{bmatrix}1\\1\end{bmatrix} = \dots$$

□

(3) Let

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 0 & 1 \\ 3 & -2 & 2 \end{bmatrix}$$

(a) Is the mapping $h\colon \mathbb{R}^3 \to \mathbb{R}^3$, $x \mapsto Ax$, injective, surjective, bijective?
(b) Give bases for null space, row space, column space of A.

Solution:

h is injective iff $\operatorname{Nul} A$ is trivial.

First find a row echelon form of A:

$$A \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 7 \\ 0 & 4 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & -4 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

For $A \cdot x = 0$ we now see that x_3 is a free variable. Set $x_3 = t$ a parameter in $\mathbb{R}$. Then $x_2 = 7/4\ t$ and $x_1 = 1/2\ t$. Hence

$$\operatorname{Nul} A = \operatorname{Span}(\begin{bmatrix} 1/2 \\ 7/4 \\ 1 \end{bmatrix}) \text{ has basis } (\begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}).$$

In particular $\operatorname{Nul} A$ is not 0 and h is not injective.

Further h **is surjective iff** $\operatorname{Col} A = \mathbb{R}^3$ **(the codomain of h).** Since A has only 2 pivots, $\dim \operatorname{Col} A = 2$ and h is not surjective. For a basis of $\operatorname{Col} A$ pick the pivot columns of A, that is

$$\operatorname{Col} A \text{ has basis } (\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}).$$

For a basis of $\operatorname{Row} A$ pick the non zero rows of an echelon form of A, that is

$$\operatorname{Row} A \text{ has basis } ((1, -2, 3), (0, -4, 7)).$$

Note that

$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \text{number of columns of } A - \dim \operatorname{Nul} A.$$

□

(4) Let A be the standard matrix for the rotation r of $\mathbb{R}^2$ by angle φ counterclockwise around the origin. What are the eigenvalues and eigenvectors of A? Can A be diagonalized over the reals?

Solution:

Version 1: The standard matrix of r is

$$A = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

Its characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2 - 2\lambda\cos\varphi + (\cos\varphi)^2 + (\sin\varphi)^2 = \lambda^2 - 2\lambda\cos\varphi + 1.$$

By the quadratic formula its roots are

$$\lambda_{1,2} = \cos\varphi \pm \sqrt{(\cos\varphi)^2 - 1} = \cos\varphi \pm i\sin\varphi.$$

Hence there are no real eigenvalues unless $\sin\varphi = 0$, that is, $\varphi = 0$ or π. In the first case $A = I$ and has eigenvalue 1 (multiplicitiy 2) with eigenspace $\mathbb{R}^2$. In the second case $A = -I$ and has eigenvalue -1 (multiplicitiy 2) with eigenspace $\mathbb{R}^2$.

If $\varphi \neq 0, \pi$, then A is not diagonalizable over $\mathbb{R}$ since it does not have any real eigenvalues.

Version 2 (with less computation): Rotation scales a vector $v \in \mathbb{R}^2$ only for $\varphi = 0$ in which case $r(v) = v$ or for $\varphi = \pi$ in which case $r(v) = -v$. Hence $A = I$ or $A = -I$ as above. □

(5) Diagonalize A if possible. Also compute $\det A$. Is A invertible?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution:

Characteristic polynomial: Expand the determinant by row 3 to get

$$\det(A - \lambda I) = (-2 - \lambda) \det \begin{bmatrix} 1-\lambda & 2 \\ 3 & -\lambda \end{bmatrix} = (-2 - \lambda)(\lambda^2 - \lambda - 6)$$

Eigenvalues: We see one root $\lambda_1 = -2$ of the characteristic polynomial and compute the others with the quadratic formula:

$$\lambda_{2,3} = 1/2 \pm \sqrt{1/4 + 6} = 1/2 \pm 5/2.$$

Hence $\lambda_2 = -2, \lambda_3 = 3$.

Eigenvectors: For $\mathrm{Nul}(A - (-2)I)$ consider

$$A + 2I = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\mathrm{Nul}(A - (-2)I)$ has basis vector $v_1 = \left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right], v_2 = \left[\begin{smallmatrix} -2 \\ 3 \\ 0 \end{smallmatrix}\right]$. For $\mathrm{Nul}(A - 3I)$ consider

$$A - 3I = \begin{bmatrix} -2 & 2 & 3 \\ 3 & -3 & 3 \\ 0 & 0 & -5 \end{bmatrix} \sim \begin{bmatrix} -2 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $\mathrm{Nul}(A - 3I)$ has basis vector $v_3 = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$.

Diagonalization of A: Since we have found 3 linear independent eigenvectors v_1, v_2, v_3, A is diagonalizable. For

$$P = [v_1 v_2 v_3], \; A = P \; \mathrm{diag}(-2, -2, 3) \; P^{-1}.$$

det$A$ can be computed with the Rule of Sarrus, row reduction, expansion by row 3 or as the product of eigenvalues (see HW 12.4):

$$\det A = \lambda_1 \lambda_2 \lambda_3 = 12.$$

A **has an inverse** since $\det A \neq 0$. □

(6) Compute the inverse if possible:

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Solution:

$$A^{-1} = \frac{1}{4-4}\begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

does not exist since $\det A = 0$.

Row reduce $[B, I_3]$:

$$\begin{bmatrix} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & -3 & 4 & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 0 & 0 & 1 \\ 0 & 1 & -4/3 & 1/3 & 0 & -1/3 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 1 & 4 & 0 \\ 0 & 1 & 0 & 1/3 & 4/3 & -1/3 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 & 4/3 & 2/3 \\ 0 & 1 & 0 & 1/3 & 4/3 & -1/3 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

So

$$B^{-1} = \begin{bmatrix} 1/3 & 4/3 & 2/3 \\ 1/3 & 4/3 & -1/3 \\ 0 & -1 & 0 \end{bmatrix}.$$

□

(7) Let $h\colon V \to W$ be a linear map, let $v_1, \ldots, v_k \in V$ such that $h(v_1), \ldots, h(v_k)$ are linearly independent. Show that $v_1, \ldots, v_k$ are linearly independent.

Solution:

Consider a linear combination

$$c_1 v_1 + \cdots + c_k v_k = 0$$

for scalars $c_1, \ldots, c_k$. We want to show that all c_i are 0.

Apply h to the equation above,

$$h(c_1 v_1 + \cdots + c_k v_k) = h(0)$$

By linearity of h this yields

$$c_1 h(v_1) + \cdots + c_k h(v_k) = 0.$$

Since $h(v_1), \ldots, h(v_k)$ are linearly independent, this implies $c_1 = \cdots = c_k = 0$. Hence the original vectors $v_1, \ldots, v_k$ are linearly independent. □