Math 2135 - Assignment 13

Due December 9, 2024

(1) (a) Give 3 vectors of length 1 in $\mathbb{R}^3$ that are orthogonal to $u = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$.

(b) Which of the following sets are orthogonal? Orthonormal?

$$A = \{\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}\}, \qquad B = \{\frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{18}}\begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}\}$$

(2) (a) Let W be the subspace of $\mathbb{R}^3$ with orthonormal basis $B = (\frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix})$. Compute the coordinates $[x]_B$ for $x = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}$ in W using dot products.

(b) Give a basis for $W^\perp$.

(c) Find the closest point to $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in W. What is the distance from y to W?

(3) True or false. Explain your answers.

(a) Every orthogonal set is also orthonormal.

(b) Not every orthonormal set in $\mathbb{R}^n$ is linearly independent.

(c) For each x and each subspace W, the vector $x - \text{proj}_W(x)$ is orthogonal to W.

(4) Let W be a subset of $\mathbb{R}^n$. Show that its orthogonal complement

$$W^\perp := \{x \in \mathbb{R}^n \mid x \text{ is orthogonal to all } w \in W\}$$

is a subspace of $\mathbb{R}^n$.

(5) Let W be a subspace of $\mathbb{R}^n$. Show that

(a) $W \cap W^\perp = 0$

(b) $\dim W + \dim W^\perp = n$

Hint: Let $w_1, \ldots, w_k$ be a basis of W. Use that $x \in W^\perp$ iff x is orthogonal to $w_1, \ldots, w_k$.

(6) Find the least squares solutions of $Ax = b$.

(a) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$

(7) True or false for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Explain your answers.

(a) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x}$ is as close as possible to b.

(b) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x} = \hat{b}$ for $\hat{b}$ the orthogonal projection of b onto $\text{Col}\, A$.

(c) The point in $\text{Col}\, A$ closest to b is a least squares solution of $Ax = b$.

(d) If $Ax = b$ is consistent, then every solution x is a least squares solution.