Math 2135 - Assignment 12

Due Nov 22, 2024

(1) Are the matrices A, B, C, D in (5), (6), (7) of assignment 11 diagonalizable? How?
Solution:
A is not diagonalizable because its eigenvalue -3 has multiplicity 2 but the corresponding eigenspace only dimension 1.
B is diagonalizable because it has 3 distinct eigenvalues, so

$$B = P \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

C is diagonalizable because it has 2 distinct eigenvalues, so

$$C = P \begin{bmatrix} 1+\sqrt{6} & 0 \\ 0 & 1-\sqrt{6} \end{bmatrix} P^{-1} \text{ for } P = \begin{bmatrix} 2 & -2 \\ \sqrt{6} & \sqrt{6} \end{bmatrix}$$

D is not diagonalizable because its eigenvalue -3 has multiplicity 2 but the corresponding eigenspace only dimension 1. □

(2) Let A be an $n \times n$-matrix. Are the following true or false? Explain why:
(a) If A has n eigenvectors, then A is diagonalizable.
(b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
(c) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
(d) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
(e) Every triangular matrix is diagonalizable.
Solution:
(a) **False.** You need n linearly independent eigenvectors.
(b) **True.**
(c) **False.** See example A in the previous problem.
(d) **True.** A basis of $\mathbb{R}^n$ of eigenvectors consists of n linearly independent eigenvectors.
(e) **False.** See example A in the previous problem. □

(3) Let A be the standard matrix for the reflection t of $\mathbb{R}^2$ on some line g throught the origin. What are the eigenvalues, eigenvectors and eigenspaces of A? Can A be diagonalized?
Hint: Consider what a reflection does to specific vectors.
Solution:
Let v_1 be a non-zero vector on the line g, that is, v_1 spans g. Then $t(v_1) = Av_1 = v_1$. Hence v_1 is an eigenvector for A (equivalently for t) with eigenvalue 1.

Let v_w be a non-zero vector orhogonal to g. Then $t(v_2) = Av_2 = -v_2$. Hence v_2 is an eigenvector for A (equivalently for t) with eigenvalue -1.

Since A is a 2×2-matrix and has at most 2 eigenvalues we found all of them. Since v_1 and v_2 are non-zero and orthogonal, they form a basis $B = (v_1, v_2)$ of $\mathbb{R}^2$. For P

the matrix with columns v_1, v_2, we then have

$$A = P \cdot \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot P^{-1}.$$

Note that P is the change of coordinates matrix $[id]_{B,E}$ and $[t]_{B \leftarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. So that's exactly how we computed $[t]_{E \leftarrow E} = A$ earlier. □

(4) As the previous problem for a rotation r of $\mathbb{R}^2$ by an angle φ around the origin.
Hint: Consider $\varphi = 0, \pi$ separately.
Solution:
For $\varphi = 0$ the rotation is just the identity map. The standard matrix is the identity matrix and already diagonalized. So every non-zero vector in $\mathbb{R}^2$ is an eigenvector for eigenvalue 1, eigenspace $E_1 = \mathbb{R}^2$.

For $\varphi = \pi$ the rotation is just scaling every vector by -1. The standard matrix is the negative of the identity matrix and already diagonalized. So every non-zero vector in $\mathbb{R}^2$ is an eigenvector for eigenvalue -1, eigenspace $E_{-1} = \mathbb{R}^2$.

Else rotating a non-zero vector x by $\varphi \neq 0, \pi$ does not give a real scalar multiple of x. Hence r has no real eigenvalues and no real eigenvectors; r cannot be diagonalized over the reals. Note that the characteristic polynomial of the standard matrix

$$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

is $\lambda^2 - 2\cos\varphi\lambda + 1$ and has complex roots $\cos\varphi \pm i\sin\varphi$. It follows that r can be diagonalized over the complex numbers. □

(5) Consider a population of owls feeding on a population of squirrels. In month k, let o_k denote the number of owls and s_k the number of squirrels. Assume that the populations change every month as follows:

$$\begin{aligned} o_{k+1} &= 0.3o_k + 0.4s_k \\ s_{k+1} &= -0.4o_k + 1.3s_k \end{aligned}$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $x_k = \begin{bmatrix} o_k \\ s_k \end{bmatrix}$. Express the population change from x_k to x_{k+1} using a matrix A. Diagonalize A.
Solution:

$$x_{k+1} = \underbrace{\begin{bmatrix} 0.3 & 0.4 \\ -0.4 & 1.3 \end{bmatrix}}_{A} x_k$$

We diagonalize A. The characteristic equation is

$$0 = \det(A - \lambda I) = (0.3 - \lambda)(1.3 - \lambda) + 0.4^2 = \lambda^2 - 1.6\lambda + 0.55,$$

the eigenvalues are $\lambda = \frac{1}{2}(1.6 \pm \sqrt{1.6^2 - 4 \cdot 0.55}) = 0.8 \pm 0.3 \in \{0.5, 1.1\}$. We compute a basis for each eigenspace.

$$\lambda = 0.5: \quad \text{Nul}(A - 0.5I) = \text{Nul}\begin{bmatrix} -0.2 & 0.4 \\ -0.4 & 0.8 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \text{Span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}.$$

$$\lambda = 1.1: \quad \text{Nul}(A - 1.1I) = \text{Nul}\begin{bmatrix} -0.8 & 0.4 \\ -0.4 & 0.2 \end{bmatrix} = \text{Nul}\begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = \text{Span}\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\}.$$

We write the eigenvectors in a matrix P and compute P^{-1}:

$$P = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}, \qquad P^{-1} = \frac{1}{3/2}\begin{bmatrix} 1 & -1/2 \\ -1 & 2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}.$$

We obtain a diagonalization

$$A = \underbrace{\begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 0.5 & 0 \\ 0 & 1.1 \end{bmatrix}}_{D} \underbrace{\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}}_{P^{-1}}.$$

□

(6) Continue the previous problem: Let the starting population be $x_1 = \begin{bmatrix} o_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.

(a) Give an explicit formula for the populations in month $k+1$.
(b) Are the populations growing or decreasing over time? By which factor?
(c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

Solution:

(a) (2 points)

$$x_{k+1} = A^k x_1 = PD^kP^{-1}x_1 = \begin{bmatrix} 2 & 1/2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0.5^k & 0 \\ 0 & 1.1^k \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 20 \\ 100 \end{bmatrix}$$
$$= \begin{bmatrix} 60 \cdot 1.1^k - 40 \cdot 0.5^k \\ 120 \cdot 1.1^k - 20 \cdot 0.5^k \end{bmatrix}$$

(b) (2 points) Both populations are growing. For large k, the term 0.5^k can be neglected (e.g. for $k \geq 12$ we have $1.1^k \geq 3.138$ and $0.5^k \leq 0.00025$). We can approximate the populations by

$$x_{k+1} \approx \begin{bmatrix} 60 \cdot 1.1^k \\ 120 \cdot 1.1^k \end{bmatrix} = 1.1^k \begin{bmatrix} 60 \\ 120 \end{bmatrix} \quad \text{for large } k.$$

After a large number of months, both populations grow by a factor of 1.1 every month.

(c) (1 point) The populations are $x_{13} = \begin{bmatrix} 188.3 \\ 376.6 \end{bmatrix}$ after 12 months and $x_{25} = \begin{bmatrix} 591.0 \\ 1182.0 \end{bmatrix}$ after 24 months. After a large number of months, the ratio of owls to squirrels is always about 1 : 2 by the approximation formula for x_{k+1}.

□