Math 2135 - Assignment 11

Due November 15, 2021

(1) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

Solution:

$$\begin{aligned}
\det A &= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} && \text{factoring 3 from the first row} \\
&= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{bmatrix} && \text{subtracting multiples of the first row from the others} \\
&= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix} && \text{adding 5 times the second row to the third} \\
&= 3 \cdot 1 \cdot 1 \cdot (-8) = -24.
\end{aligned}$$

$$\begin{aligned}
\det B &= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{bmatrix} \\
&= -\det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{bmatrix} && \text{flipped row 3 and 4} \\
&= -1 \cdot 1 \cdot 1 \cdot 10 = -10.
\end{aligned}$$

□

(2) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} u & v \\ w & x \end{bmatrix}$. Show

$$\det(AB) = \det(A)\det(B).$$

Solution:

$$AB = \begin{bmatrix} au+bw & av+bx \\ cu+dw & cv+dx \end{bmatrix}$$

$$\det AB = (au+bw)(cv+dx) - (av+bx)(cu+dw) = \cdots = \det A \cdot \det B$$

□

(3) Let $A \in \mathbb{R}^{n\times n}$. Are the following true or false? Explain why:
 (a) If two rows or columns of A are identical, then $\det A = 0$.
 (b) For $c \in \mathbb{R}$, $\det(cA) = c \det A$.
 (c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.
 (d) A is invertible iff 0 is not an eigenvalue of A.

Solution:
(a) True. If two rows or columns of A are identical, then A is not invertible and $\det A = 0$.
(b) False. $\det(cA) = c^n \det A$ since in cA every row is multiplied by c.
(c) True. Assume A is invertible. Then $\det A \cdot A^{-1} = \det A \cdot \det A^{-1}$ by a Theorem from class. Since $\det A \cdot A^{-1} = \det I = 1$, the statement follows.
(d) True. By the Invertible Matrix Theorem A is invertible iff $\operatorname{Nul} A$ is trivial. The latter means that $\operatorname{Nul}(A - 0I) = \{0\}$, i.e. 0 is not an eigenvalue of A. □

(4) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h\colon V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

Solution:
We have to show that $E_{h,\lambda}$ contains the 0-vector, is closed under addition and scalar multiples. Using the linearity of h we get:
$0 \in E_{h,\lambda}$ since $h(0) = 0 = \lambda 0$
If $u, v \in E_{h,\lambda}$, then $h(u+v) = h(u) + h(v) = \lambda u + \lambda v = \lambda(u+v)$ and $u + v \in E_{h,\lambda}$.
If $v \in E_{h,\lambda}$ and $c \in F$, then $h(cv) = ch(v) = c\lambda v = \lambda cv$ and $cv \in E_{h,\lambda}$. □

(5) Give all eigenvalues and bases for eigenspaces of the following matrices. Do you need the characteristic polynomials?

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

Solution:
Since A, B are triangular matrices, their eigenvalues are just their diagonal elements.
 (a) A has eigenvalue -3 with multiplicity 2: $\operatorname{Nul}(A - (-3)I)$ has basis $(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$
 (b) B has eigenvalues $2, 0, 3$:

Nul$(A-2I)$ has basis $\left(\begin{bmatrix}2\\1\\2\end{bmatrix}\right)$.
Nul$(A-0I)$ has basis $\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)$.
Nul$(A-3I)$ has basis $\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$.

□

(6) Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix}1 & 2\\3 & 1\end{bmatrix}$.

Solution:
Characteristic polynomial:

$$\begin{aligned}\det(C-\lambda I) &= (1-\lambda)(1-\lambda) - 2\cdot 3\\ &= \lambda^2 - 2\lambda - 5\end{aligned}$$

Eigenvalues are the roots of the characteristic polynomial. Use the quadratic formula

$$\begin{aligned}\lambda_{1,2} &= 1 \pm \sqrt{1+5}\\ &= 1 \pm \sqrt{6}\end{aligned}$$

Eigenvector for $\lambda = 1+\sqrt{6}$:

$$C - \lambda I = \begin{bmatrix}-\sqrt{6} & 2\\ 3 & -\sqrt{6}\end{bmatrix} \sim \begin{bmatrix}-\sqrt{6} & 2\\ 0 & 0\end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and added to row 2.

So the eigenspace for $\lambda = 1+\sqrt{6}$ has basis $\left(\begin{bmatrix}2\\ \sqrt{6}\end{bmatrix}\right)$.

Eigenvector for $\lambda = 1-\sqrt{6}$:

$$C - \lambda I = \begin{bmatrix}\sqrt{6} & 2\\ 3 & \sqrt{6}\end{bmatrix} \sim \begin{bmatrix}\sqrt{6} & 2\\ 0 & 0\end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and subtracted from row 2.

So the eigenspace for $\lambda = 1-\sqrt{6}$ has basis $\left(\begin{bmatrix}-2\\ \sqrt{6}\end{bmatrix}\right)$.

□

(7) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix}-1 & 4 & 1\\ 6 & 9 & 2\\ 0 & 0 & -3\end{bmatrix}$.

Solution:
Characteristic polynomial:

$$\begin{aligned}\det(D-\lambda I) &= (-3-\lambda)\cdot\det\begin{bmatrix}-1-\lambda & 4\\ 6 & 9-\lambda\end{bmatrix}\\ &(-3-\lambda)[(-1-\lambda)(9-\lambda) - 24]\\ &(-3-\lambda)[\lambda^2 - 8\lambda - 33]\end{aligned}$$

Eigenvalues are $\lambda_1 = -3$ and the roots of $\lambda^2 - 8\lambda - 33$. The quadratic formula yields

$$\lambda_{2,3} = 4 \pm \sqrt{4^2 + 33}$$

So $\lambda_2 = -3$ and $\lambda_3 = 11$.

The eigenspace for $\lambda = -3$ has basis $\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\right)$.

The eigenspace for $\lambda = 11$ has basis $\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right)$. $\square$