Math 2135 - Assignment 10

Due November 8, 2024

Problems 1-5 are review material for the second midterm on November 6. Solve them before Wednesday!

(1) Let V, W be vector spaces over $\mathbb{R}$ with zero vectors $0_V, 0_W$, respectively. Let $f: V \to W$ be linear. Show
 (a) $f(0_V) = 0_W$,
 (b) the kernel $\ker f := \{v \in V \;:\; f(v) = 0_W\}$ of f is a subspace of V.

Solution:
(a) By linearity $f(0_V) = f(0_V + 0_V) = f(0_V) + f(0_V)$. Subtracting $f(0_V)$ from both sides yields $0_W = f(0_v)$.
(b) $\ker f$ contains 0_V by (a).
Show $\ker f$ is closed under addition: Let $u, v \in \ker f$, that is, $f(u) = 0, f(v) = 0$. By linearity $f(u+v) = f(u) + f(v) = 0 + 0 = 0$. Hence $u + v \in \ker f$.
Show $\ker f$ is closed under scalar multiples: Let $c \in \mathbb{R}$ and $v \in \ker f$, that is, $f(v) = 0$. By linearity $f(cv) = cf(v) = c0 = 0$. Hence $cv \in \ker f$. □

(2) Let $T: P_2 \to \mathbb{R}, p \mapsto p(3)$, be the map that evaluates a polynomial p at $x = 3$.
 (a) Show that T is linear.
 (b) Determine the kernel of T, that is, $\ker T = \{p \in P_2 \;:\; T(p) = 0\}$, and the image of T, that is, $T(P_2)$.
 (c) Is T injective, surjective, bijective?

Solution:

 (a) For linearity, let $p, q \in P_2$. Their sum $p + q$ is the polynomial that maps t to $p(t) + q(t)$. So
$$T(p + q) = (p + q)(3) = p(3) + q(3) = T(p) + T(q).$$
Further let $c \in \mathbb{R}$. Then cp maps t to $cp(t)$. So
$$T(cp) = (cp)(3) = cp(3) = cT(p).$$
Hence T is linear.
 (b) The kernel of T, $\ker T$, consists of all the polynomials that evaluate to 0 at 3, that is,
$$\ker T = \{(t - 3)q : q \in P_1\}.$$
The range of T, $T(P_2)$, is $\mathbb{R}$. For every $b \in \mathbb{R}$, there exists a polynomial $p \in P_2$ that is mapped to b. Choose for example the constant polynomial $p(t) = b$.
 (c) Since the kernel of T is non-trivial, T is not injective.
Since the range of T is equal to its codomain, T is surjective.
T is not bijective since it is not injective.

□

(3) Let $B = (b_1, b_2)$ with $b_1 = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $C = \left(\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}\right)$ be bases of a subspace H of $\mathbb{R}^3$.
 (a) Compute the coordinates $[b_1]_C$ and $[b_2]_C$.

(b) What is the change of coordinate matrix $P_{C \leftarrow B}$?
(c) What is the change of coordinate matrix $P_{B \leftarrow C}$?

Solution:
(a) Solve the linear system

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}$$

to obtain $x_1 = 3, x_2 = -4$. So $[b_1]_C = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

Similarly we get $[b_2]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Since the columns of $P_{C \leftarrow B}$ are just the coordinate tuples $[b_i]_C$, we see

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$$

(c)

$$P_{B \leftarrow C} = P_{C \leftarrow B}^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$$

Alternatively we could also get $P_{B \leftarrow C}$ from its columns $[c_i]_B$. □

(4) Let $C = (1+t, t+t^2, 1+t^2)$ be a basis for P_2. Compute the coordinates $[p]_C$ for $p = 2 + t^2$.

Solution:
Solve

$$c_1(1+t) + c_2(t+t^2) + c_3(1+t^2) = 2 + t^2.$$

Comparing the coefficients on both sides of this equation yields

$$\begin{aligned} c_1 + c_3 &= 2 && \text{(constant part)} \\ c_1 + c_2 &= 0 && \text{(multiples of } t\text{)} \\ c_2 + c_3 &= 1 && \text{(multiples of } t^2\text{)} \end{aligned}$$

Solving that system of linear equations yields $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}, c_3 = \frac{3}{2}$. So $[u]_B = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$. □

(5) (a) Show that $A \in \mathbb{R}^{n \times n}$ is invertible iff rank $A = n$.

Solution:
$A \in \mathbb{R}^{n \times n}$ is invertible
iff the columns of A form a basis of $\mathbb{R}^n$ by the Inverse Matrix Theorem
iff rank $A = n$ by the definition of rank as the dimension of Col A. □

(b) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of Nul A?

Solution:
The rank of a matrix is the number of its pivot elements, which is at most the number of its rows and at most the number of its columns. So rank$A \leq$

$\max(3,4) = 3$. Since the largest possible rank is 3, the smallest number of free variables in $Ax = 0$ is 1. So the dimension of Nul A is 1 or larger. □

(c) If the nullspace of a 4×6-matrix B has dimension 3, what is the dimension of the row space of B?

Solution:
dim Nul $A +$ dim Row $A =$ the number of columns of A
So dim Row $A = 3$. □

(6) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

Solution:
Expand $\det A$ down the first column:

$$\det A = 0 \cdot \det A_{11} - 5 \cdot \det A_{21} + 0 \cdot \det A_{31} = -5 \cdot \det \begin{bmatrix} 1 & -3 \\ -3 & -4 \end{bmatrix} = -5(1(-4) - (-3)(-3)) = 65$$

Expand $\det B$ across 3rd row:

$$\det B = 2 \cdot \det B_{13} = 2 \cdot \det \begin{bmatrix} 0 & -3 & 0 \\ 1 & 5 & 1 \\ 1 & -2 & 5 \end{bmatrix}$$

Expand across 1st row:

$$\det B_{13} = -1(-3) \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 3 \cdot (1 \cdot 5 - 1 \cdot 1) = 12$$

So $\det B = 2 \cdot 12 = 24$. □

(7) **Rule of Sarrus for the determinant of** 3×3**-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

Solution:

$$\begin{aligned}\det A &= a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13} \\ &= a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

□

(8) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

Solution:
Interchanging 2 rows changes the sign of the determinant:

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = -\det A$$

□

(b) How does multiplying one row by a scalar effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

(c) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix}$.

Solution:
Adding a multiple of the first row to another does not change the determinant:

$$\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix} = a(d + rb) - b(c + ra) = ad - bc = \det A$$

□