Math 2135 - Assignment 10

Due November 8, 2024

Problems 1-5 are review material for the second midterm on November 6. Solve them before Wednesday!

(1) Let V, W be vector spaces over $\mathbb{R}$ with zero vectors $0_V, 0_W$, respectively. Let $f\colon V \to W$ be linear. Show
 (a) $f(0_V) = 0_W$,
 (b) the kernel $\ker f := \{v \in V \ : \ f(v) = 0_W\}$ of f is a subspace of V.

(2) Let $T\colon P_2 \to \mathbb{R}, p \mapsto p(3)$, be the map that evaluates a polynomial p at $x = 3$.
 (a) Show that T is linear.
 (b) Determine the kernel of T, that is, $\ker T = \{p \in P_2 \ : \ T(p) = 0\}$, and the image of T, that is, $T(P_2)$.
 (c) Is T injective, surjective, bijective?

(3) Let $B = (b_1, b_2)$ with $b_1 = \left[\begin{smallmatrix} -5 \\ 11 \\ 5 \end{smallmatrix}\right], b_2 = \left[\begin{smallmatrix} 3 \\ -1 \\ 4 \end{smallmatrix}\right]$ and $C = (\left[\begin{smallmatrix} 1 \\ 1 \\ 3 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ -2 \\ 1 \end{smallmatrix}\right])$ be bases of a subspace H of $\mathbb{R}^3$.
 (a) Compute the coordinates $[b_1]_C$ and $[b_2]_C$.
 (b) What is the change of coordinate matrix $P_{C\leftarrow B}$?
 (c) What is the change of coordinate matrix $P_{B\leftarrow C}$?

(4) Let $C = (1 + t, t + t^2, 1 + t^2)$ be a basis for P_2. Compute the coordinates $[p]_C$ for $p = 2 + t^2$.

(5) (a) Show that $A \in \mathbb{R}^{n\times n}$ is invertible iff $\operatorname{rank} A = n$.
 (b) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of Nul A?
 (c) If the nullspace of a 4×6-matrix B has dimension 3, what is the dimension of the row space of B?

(6) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

(7) **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

(8) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

(b) How does multiplying one row by a scalar effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

(c) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix}$.