Math 2135 - Assignment 9

Due November 1, 2024

(1) Let $b_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 2.5 \\ -5 \end{bmatrix}$.

(a) Find vectors $u_1, \ldots, u_k$ such that $(b_1, b_2, u_1, \ldots, u_k)$ is a basis for $\mathbb{R}^3$.

(b) Find vectors $v_1, \ldots, v_\ell$ such that $(b_3, v_1, \ldots, v_\ell)$ is a basis for $\mathbb{R}^3$.

Prove that your choices for (a) and (b) form a basis.

Solution:

Both bases have 3 vectors. Thus $k = 1$ and $\ell = 2$.

(a) One possible choice is $u_1 = e_1$. We show that (b_1, b_2, e_1) is a basis by reducing the following augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

(b) One possible choice is $v_1 = e_1$ and $v_2 = e_2$. We show that (b_3, e_1, e_2) is a basis by reducing the following augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2.5 & 0 & 1 & 0 \\ -5 & 0 & 0 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

There is no free variable. Thus the columns are linearly independent. Also, the vectors span $\mathbb{R}^3$ since there is no zero row.

□

(2) A 25×35 matrix A has 20 pivots. Find $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.

Solution:

The number of pivots, $\dim \operatorname{Row} A$, $\dim \operatorname{Col} A$, and the rank are equal. So

$$\dim \operatorname{Row} A = \dim \operatorname{Col} A = \operatorname{rank} A = 20.$$

By the rank theorem, $\dim \operatorname{Nul} A + \operatorname{rank} A = 35$. Thus

$$\dim \operatorname{Nul} A = 35 - 20 = 15.$$

□

(3) True or false? Explain.

(a) A basis of B is a set of linear independent vectors in V that is as large as possible.

(b) If $\dim V = n$, then any n vectors that span V are linearly independent.

(c) Every 2-dimensional subspace of $\mathbb{R}^2$ is a plane.

Solution:

(a) True. If linear independent vectors $a_1, \ldots, a_k$ in V do not span V yet, you can get a bigger linear set by adding a vector a_{k+1} which is not in $\operatorname{Span}\{a_1, \ldots, a_k\}$.

(b) True by the Basis Theorem.

(c) True since any two linear independent vectors in $\mathbb{R}^2$ span a plane through the origin.

$\square$

(4) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.
(a) Find the matrix $d_{B\leftarrow B}$ for the derivation map $d\colon P_3 \to P_3, p \to p'$.
(b) Use $d_{B\leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.
Solution:
Compute coordinates of the derivatives $d(b_i)$ for the basis vectors in B to get

$$d_{B\leftarrow B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $[p']_B = d_{B\leftarrow B}[p]_b = (2, 0, 3)$ and $p' = 2 + 3x^2$. $\square$

(5) Let $B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $C = (\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.
(a) Find the change of coordinates matrix $P_{E\leftarrow B}$ for $f : [u]_B \mapsto [u]_E$.
(b) Find the change of coordinates matrix $P_{C\leftarrow E}$ for $g : [u]_E \mapsto [u]_C$.
(c) Find the change of coordinates matrix $P_{C\leftarrow B}$ for $h : [u]_B \mapsto [u]_C$.
Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$.
Solution:
Let E be the standard basis of $\mathbb{R}^2$.
(a) How to compute E-coordinates from B-coordinates? The standard matrix for f is

$$P_{B\leftarrow E} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Note that the columns are exactly the vectors of B. Changing coordinates from any B to the standard basis E is easy.
(b) How to compute C-coordinates from E-coordinates? The standard matrix for g is

$$P_{E\leftarrow C} = P_{C\leftarrow E}^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

Note that the matrix is the inverse of the matrix whose columns are the vectors of C. For changing coordinates from the standard basis E to a basis C you need to solve a linear system or find the inverse.
(c) How to compute C-coordinates from B-coordinates? First go from B-coordinates to E-coordinates and then to C-coordinates. The matrix for $h = g \circ f$ is

$$P_{C\Leftarrow B} = P_{C\leftarrow E}P_{E\Leftarrow B} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$$

$\square$

(6) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:

(a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.

(b) Give the matrix $t_{B\leftarrow B}$ for the reflection with respect to the coordinate system determined by B.

(c) Use the change of coordinate matrix to compute the standard matrix $t_{E\leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

Solution:

(a) Pick $B = (\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix})$ with the first vector b_1 on the line $3x + y = 0$, the second b_2 orthogonal. Then $t(b_1) = b_1, t(b_2) = -b_2$.

(b)

$$t_{B\leftarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

looks like the standard matrix for the reflection on the x-axis.

(c) To get $t_{E\leftarrow E}$ from $[t]_{B,B}$ we need to multiply with change of coordinate matrices,

$$t_{E\leftarrow E} = P_{B\leftarrow E} t_{B\leftarrow B} P_{E\leftarrow B} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 \\ -3 & -1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -8 & -6 \\ -6 & 8 \end{bmatrix}$$

□

(7) (a) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?

(b) Consider the rotation s of $\mathbb{R}^3$ around the line spanned by $\left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ through the angle $\pi/3$ counterclockwise. Find a basis of $\mathbb{R}^3$ for which the matrix $s_{B\leftarrow B}$ is equal to A from (a).

(c) Give the standard matrix $s_{E\leftarrow E}$ for the standard basis E (You do not need to actually multiply and invert the involved matrices; the product formula is enough).

Solution:

(a) e_3 remains fixed, e_1, e_2 rotate like in $\mathbb{R}^2$, i.e.,

$$A = \begin{bmatrix} \cos \pi/3 & -\sin \pi/3 & 0 \\ \sin \pi/3 & \cos \pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We want $b_3 = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ (fixed under rotation) and b_1, b_2 in a plane orthogonal to b_3, orthogonal to each other and of length 1, e.g.,

$$b_1 = \frac{1}{\sqrt{5}} \left[\begin{smallmatrix} 2 \\ -1 \\ 0 \end{smallmatrix}\right], b_2 = \frac{1}{\sqrt{70}} \left[\begin{smallmatrix} 3 \\ 6 \\ -5 \end{smallmatrix}\right] \text{ (the normalized vector for } b_3 \times b_1\text{).}$$

Note that the basis b_1, b_2, b_3 is right-handed since $b_1 \times b_2$ points in direction of b_3. For $B = (b_1, b_2, b_3)$ the matrix $s_{B\leftarrow B}$ is equal to A from (a).

(c) To get the standard matrix $s_{E\leftarrow E}$ from $s_{B\leftarrow B}$ we need to multiply with change of coordinate matrices: let

$$P_{E\leftarrow B} = [b_1, b_2, b_3] =: P$$

be the matrix with vectors b_1, b_2, b_3 in its columns. Then

$$s_{E\leftarrow E} = P_{E\leftarrow B} s_{B,B} P_{E\leftarrow B} = P \cdot A \cdot P^{-1}$$

□

(8) The *kernel* of a linear map $h\colon V \to W$ is the subspace of V,

$$\ker h := \{v \in V \mid h(v) = 0\}.$$

(a) Determine the kernel and the image of $d\colon P_3 \to P_3, p \to p'$.
(b) Is d injective, surjective, bijective?

Solution:

(a) $p \in \ker d$ iff $p' = 0$ iff p is a constant polynomial iff p has degree ≤ 0. Hence

$$\ker d = P_0.$$

For $p \in P_3$ we see that p' has degree 2, hence $p' \in P_2$. So $d(P_3) \subseteq P_2$.
We claim that conversely every $q = a_0 + a_1 x + a_2 x^2 \in P_2$ is in $d(P_3)$, that is the derivative of a polynomial p of degree 3. Note that $p = a_0 x + \frac{a_1}{2} x^2 + \frac{a_2}{3} x^3$ satisfies $p' = q$. Thus

$$d(P_3) = P_2.$$

(b) d is not injective since $\ker d \neq 0$.
d is not surjective since $d(P_3) = P_2 \neq P_3$.
Hence d is not bijective.

Alternatively for (b) one can also compute the nullspace of the matrix $d_{B\leftarrow B}$ to see that $\ker d$ is not trivial and its column space to see that $d(P_3) \neq P_3$. But for (a) you'd need to translate $\operatorname{Nul} d_{B\leftarrow B}$ back to P_0 and $\operatorname{Col} d_{B\leftarrow B}$ back to P_2. □