Math 2135 - Assignment 9

Due November 1, 2024

(1) Let $b_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $b_3 = \begin{bmatrix} 1 \\ 2.5 \\ -5 \end{bmatrix}$.

  (a) Find vectors $u_1, \dots, u_k$ such that $(b_1, b_2, u_1, \dots, u_k)$ is a basis for $\mathbb{R}^3$.

  (b) Find vectors $v_1, \dots, v_\ell$ such that $(b_3, v_1, \dots, v_\ell)$ is a basis for $\mathbb{R}^3$.

  Prove that your choices for (a) and (b) form a basis.

(2) A 25×35 matrix A has 20 pivots. Find $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$, $\dim \operatorname{Row} A$, and $\operatorname{rank} A$.

(3) True or false? Explain.

  (a) A basis of B is a set of linear independent vectors in V that is as large as possible.

  (b) If $\dim V = n$, then any n vectors that span V are linearly independent.

  (c) Every 2-dimensional subspace of $\mathbb{R}^2$ is a plane.

(4) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.

  (a) Find the matrix $d_{B \leftarrow B}$ for the derivation map $d \colon P_3 \to P_3, p \to p'$.

  (b) Use $d_{B \leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.

(5) Let $B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $C = (\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.

  (a) Find the change of coordinates matrix $P_{E \leftarrow B}$ for $f : [u]_B \mapsto [u]_E$.

  (b) Find the change of coordinates matrix $P_{C \leftarrow E}$ for $g : [u]_E \mapsto [u]_C$.

  (c) Find the change of coordinates matrix $P_{C \leftarrow B}$ for $h : [u]_B \mapsto [u]_C$.
  Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$.

(6) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:

  (a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.

  (b) Give the matrix $t_{B \leftarrow B}$ for the reflection with respect to the coordinate system determined by B.

  (c) Use the change of coordinate matrix to compute the standard matrix $t_{E \leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

(7) (a) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
  Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?

  (b) Consider the rotation s of $\mathbb{R}^3$ around the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ through the angle $\pi/3$ counterclockwise. Find a basis of $\mathbb{R}^3$ for which the matrix $s_{B \leftarrow B}$ is equal to A from (a).

  (c) Give the standard matrix $s_{E \leftarrow E}$ for the standard basis E (You do not need to actually multiply and invert the involved matrices; the product formula is enough).

(8) The *kernel* of a linear map $h \colon V \to W$ is the subspace of V,

$$\{v \in V \mid h(v) = 0\}.$$

  (a) Determine the kernel and the image of $d \colon P_3 \to P_3, p \to p'$.

  (b) Is d injective, surjective, bijective?