Math 2135 - Assignment 8

Due October 25, 2024

(1) Show that the vectors $\mathbf{v}_0 = 1$, $\mathbf{v}_1 = 1 + t$, $\mathbf{v}_2 = 1 + t + t^2$ form a basis for the vector space P_2 of polynomials of degree ≤ 2.
Solution:
For the linear independence of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ consider the equation

$$x_0 + x_1(1 + t) + x_2(1 + t + t^2) = 0$$

at distinct values for t, e.g., $t = 0, 1, 2$ to obtain the linear system

$$\begin{aligned} x_0 + x_1 + x_2 &= 0 \\ x_0 + 2x_1 + 3x_2 &= 0 \\ x_0 + 3x_1 + 7x_2 &= 0 \end{aligned}$$

This only has the trivial solution $x_0 = x_1 = x_2 = 0$. So $1, t, t^2$ are linearly independent.

To see that $\mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2) = P_2$ show that for every polynomial $a+bt+ct^2 \in P_2$

$$x_0 + x_1(1 + t) + x_2(1 + t + t^2) = a + bt + ct^2$$

has solutions $x_0, x_1, x_2 \in \mathbb{R}$. For this we compare the coefficients of the polynomials in that equation and try to solve the resulting linear system:

$$\begin{aligned} \text{constants: } & x_0 + x_1 + x_2 = a \\ \text{coefficient of } t: & \quad x_1 + x_2 = b \\ \text{coefficient of } t^2: & \quad x_2 = c \end{aligned}$$

This has the solution $x_2 = c, x_1 = b - c, x_0 = a - b$. Hence any polynomial $a + bt + ct^2$ is in $\mathrm{Span}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$.

Since $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ is linear independent and spans P_2, it is a basis of P_2. □

(2) Give a basis for $\mathrm{Nul}(A)$ and a basis for $\mathrm{Col}(A)$ for

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 1 & -4 & -1 & 0 \\ -2 & 6 & 2 & -3 \end{bmatrix}$$

Solution:
Nul A is the solution set of $Ax = 0$. So we row reduce A

$$A = \begin{bmatrix} 0 & 2 & 0 & 3 \\ 1 & -4 & -1 & 0 \\ -2 & 6 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & -2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -1 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

to get the solution $x_4 = t, x_3 = s$ (both free), $x_2 = -\frac{3}{2}t, x_1 = s - 6t$. So

$$x = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \text{ and Nul } A \text{ has basis } \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right)$$

For a basis of the column space Col$A$ we pick the pivot columns of A, i.e., the first and second column. So Col$A$ has basis $\left(\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}\right)$. □

(3) Give 2 different bases for

$$H = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}\right\}$$

Solution:
Row reduction yields

$$\begin{bmatrix} 1 & 3 & -1 \\ 1 & -1 & 3 \\ 2 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the first 2 columns of the original matrix are pivot columns and form a basis of H.

$$B_1 = \left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}\right)$$

Recall that the order of basis vectors is important (coordinates!). So by flipping the vectors we get a different basis

$$B_2 = \left(\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}\right)$$

Also note that any 2 columns of the matrix are linearly independent. So any 2 distinct vectors in any order out of $\left\{\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}\right\}$ form a basis. □

(4) Let $B = (b_1, \ldots, b_n)$ be a basis for a vector space V and consider the coordinate mapping $V \to \mathbb{R}^n$, $x \mapsto [x]_B$.
   (a) Show that $[c \cdot x]_B = c[x]_B$ for all $x \in V, c \in \mathbb{R}$.
   (b) Show that the coordinate mapping is onto $\mathbb{R}^n$.

Solution:

(a) Let $x \in V$ with $x = c_1 b_1 + \ldots c_n b_n$ for $c_1, \ldots, c_n \in \mathbb{R}$. That is, $[x]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Let $c \in \mathbb{R}$ and consider

$$cx = c(c_1 b_1 + \ldots c_n b_n) = cc_1 b_1 + \ldots cc_n b_n.$$

Then the coordinates of cx are

$$[cx]_B = \begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix} = c \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = c[x]_B.$$

(b) To show the map is onto $\mathbb{R}^n$, let $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$. We have to find $x \in V$ such that $[x]_B = y$. Pick $x = y_1 b_1 + \dots y_n b_n$. This shows that the coordinate map is onto. □

(5) Let $B = (\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix})$ be a basis of $\mathbb{R}^2$.

(a) Find vectors $u, v \in \mathbb{R}^2$ with $[u]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [v]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

(b) Compute the coordinates relative to B of $w = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Solution:

(a) Put the vectors of B in the columns of a matrix,

$$P_{E \leftarrow B} = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$$

$u = P_{E \leftarrow B} \cdot [u]_B = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, the second vector in B

$v = \begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

(b) To find $[w]_B$ we can solve $P_{E \leftarrow B} \cdot [w]_B = w$ directly by row reduction. Alternatively, we can invert $P_{E \leftarrow B}$ and use the formula $[w]_B = P_{E \leftarrow B}^{-1} \cdot w$.

$$P_{E \leftarrow B}^{-1} = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

So

$$[w]_B = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$[x]_B = \frac{1}{-2} \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

□

(6) Let $B = (1, t, t^2)$ and $C = (1, 1 + t, 1 + t + t^2)$ be bases of $\mathbb{P}_2$.

(a) Determine the polynomials p, q with $[p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ and $[q]_C = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

(b) Compute $[r]_B$ and $[r]_C$ for $r = 3 + 2t + t^2$.

Solution:

(a) $p = 3 - 2t^2$,
$q = 3 \cdot 1 + 0 \cdot (1 + t) - 2(1 + t + t^2) = 1 - 2t - 2t^2$

(b) For the coordinates relative to B just take the coefficients of the polynomial:

$[r]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

For the coordinates relative to C consider the equation

$$\begin{aligned} r &= x_1 \cdot 1 + x_2(1+t) + x_3(1+t+t^2) \\ &= (x_1+x_2+x_3) + (x_2+x_3)t + x_3t^2 \end{aligned}$$

Comparing the coefficients we obtain $x_3 = 1, x_2 = 1, x_1 = 1$. So $[r]_C = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ □

(7) Let

$$A = \begin{bmatrix} -5 & 8 & 0 & -17 & -2 \\ 3 & -5 & 1 & 5 & 1 \\ 11 & -19 & 7 & 1 & 3 \\ 7 & -13 & 5 & -3 & 1 \end{bmatrix}.$$

Find bases and dimensions for Nul A, Col A, and Row A, respectively.

Solution:

We reduce A to reduced echelon form:

$$A \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 5 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

For Nul A, we solve $A\mathbf{x} = \mathbf{0}$ and obtain

$$\operatorname{Nul} A = \{r \begin{bmatrix} -5 \\ -1 \\ 5 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \mid r, s \in \mathbb{R}\}.$$

The two vectors form a basis for Nul A.

The first three columns of A contain a pivot. Thus they form a basis

$$B = (\begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 8 \\ -5 \\ -19 \\ -13 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix})$$

for Col A.

The nonzero rows in any echelon form of A form a basis. E.g.,

$$C = (\begin{bmatrix} 1 \\ 0 \\ 0 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -5 \\ 0 \end{bmatrix})$$

is a basis for Row A. □

(8) True or false? Explain.

(a) If B is an echelon form of a matrix A, then the pivot columns of B form a basis for the column space of A.

(b) If B is an echelon form of a matrix A, then the nonzero rows of B form a basis for the row space of A.

Solution:

(a) False! In general the columns of B will not span $\operatorname{Col} A$ any more. The pivot columns of A form a basis for $\operatorname{Col} A$.
(b) True. The rows of B still span $\operatorname{Row} A$.

$\square$