Math 2135 - Assignment 7

Due October 18, 2024

(1) Explain why the following are not subspaces of $\mathbb{R}^2$. Give explicit counter examples for subspace properties that are not satisfied.
  (a) $U = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, x \geq 0\}$
  (b) $V = \mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of all integers)
  (c) $W = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x, y \in \mathbb{R}, |x| = |y|\}$

Solution:

(a) Not closed under scalar multiples, e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in U$ but $(-1)\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin U$
(b) Not closed under scalar multiples, e.g. $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$ but $\sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \notin V$
(c) Not closed under addition, e.g. $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in W$ but $\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \notin W$

□

(2) Which of the following are subspaces of the vector space $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$? Check all subspace properties or give one that is not satisfied.
  (a) $\{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 1\}$
  (b) $\{f : \mathbb{R} \to \mathbb{R} \mid f(3) = 0\}$
  (c) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$

Solution:

(a) No subspace since it does not contain the zero vector, i.e., the constant 0-function.
(b) Subspace since (1) contains the constant 0-function, (2) is closed under addition [for functions f, g with $f(1) = 0$ and $g(1) = 0$, also $(f + g)(1) = 0 + 0 = 0$], (3) is closed under scalar multiples [if $c \in \mathbb{R}$ and $f(1) = 0$, then also $(cf)(0) = c0 = 0$].
(c) Subspace since (1) the constant 0-function is continuous, (2) the sum of continuous functions is continuous, (3) any scalar multiple of a continuos function is continuous.

□

(3) Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be vectors in a vector space V. Show that $U := \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a subspace of V.

Solution:

We show the 3 conditions for being a subspace.
  (a) The zero vector can be written as linear combination $\mathbf{0} = 0\mathbf{v}_1 + \ldots + 0\mathbf{v}_n$. Thus $\mathbf{0} \in U$.

(b) Let $\mathbf{u}$ and $\mathbf{w}$ be arbitary vectors in U. We can write these vectors as

$$\mathbf{u} = a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n \quad \text{for some } a_1, \ldots, a_n \in \mathbb{R},$$
$$\mathbf{w} = b_1\mathbf{v}_1 + \ldots + b_n\mathbf{v}_n \quad \text{for some } b_1, \ldots, b_n \in \mathbb{R}.$$

Now

$$\begin{aligned}\mathbf{u} + \mathbf{w} &= a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n + b_1\mathbf{v}_1 + \ldots + b_n\mathbf{v}_n \\ &= (a_1 + b_1)\mathbf{v}_1 + \ldots + (a_n + b_n)\mathbf{v}_n.\end{aligned}$$

Thus $\mathbf{u} + \mathbf{w}$ is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and hence an element of U.

(c) Let $\mathbf{u} \in U$ as above, and let $r \in \mathbb{R}$. Then

$$\begin{aligned}r\mathbf{u} &= r(a_1\mathbf{v}_1 + \ldots + a_n\mathbf{v}_n) \\ &= ra_1\mathbf{v}_1 + \ldots + ra_n\mathbf{v}_n.\end{aligned}$$

Thus $r\mathbf{u}$ is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and hence an element of U.

□

(4) Let $A \in \mathbb{R}^{m\times n}$. Prove that $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

Solution:

We show the 3 conditions for being a subspace.

(a) The zero vector is clearly in $\text{Nul}(A)$ since $A\mathbf{0} = \mathbf{0}$.

(b) Let $\mathbf{u}$ and $\mathbf{w}$ be arbitary vectors in $\text{Nul}(A)$. Then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$. We show that $\mathbf{u} + \mathbf{w}$ is in $\text{Nul}(A)$.

$$A(\mathbf{u} + \mathbf{w}) = A\mathbf{u} + A\mathbf{w} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So $\mathbf{u} + \mathbf{w}$ is in $\text{Nul}(A)$.

(c) Let $r \in \mathbb{R}$. Then

$$A(r\mathbf{u}) = r(A\mathbf{u}) = r\mathbf{0} = \mathbf{0}.$$

Hence $r\mathbf{u}$ is in $\text{Nul}(A)$.

□

(5) Explain whether the following are true or false (give counter examples if possible):

(a) Every vector space is a subspace of itself.

(b) Each plane in $\mathbb{R}^3$ is a subspace.

(c) Let U be a subspace of a vector space V. Any linear combination of vectors of U is also in V.

(d) Let $v_1, \ldots, v_n$ be in a vector space V. Then $\text{Span}(v_1, \ldots, v_n)$ is the smallest subspace of V containing $v_1, \ldots, v_n$.

Solution:

(a) True, since it contains zero vector, is closed under addition and scalar multiples.

(b) False, e.g., the plane $z = 1$ does not contain the zero vector.

(c) True, since U is closed under under addition and scalar multiples by definition, U is also closed under linear combinations.

(d) True, by (c) every subspace of V containing $v_1, \ldots, v_n$ also contains $\text{Span}(v_1, \ldots, v_n)$ which is a subspace by a previous HW-problem. So $\text{Span}(v_1, \ldots, v_n)$ is the smallest subspace of V containing $v_1, \ldots, v_n$.

□

(6) Are the vectors $\mathbf{v}_0 = 1$, $\mathbf{v}_1 = t$, $\mathbf{v}_2 = t^2$ in the vector space $\mathbb{R}^\mathbb{R} := \{f : \mathbb{R} \to \mathbb{R}\}$ linearly independent?
Solution:
Yes, for real numbers x_0, x_1, x_2 the polynomial $x_0 + x_1 t + x_2 t^2$ is equal to the constant 0-function iff $x_0 = x_1 = x_2 = 0$.
Alternatively, consider the equation

$$x_0 + x_1 t + x_2 t^2 = 0$$

at distinct values for t, e.g., $t = 0, 1, 2$ to obtain the linear system

$$\begin{aligned} x_0 + 0x_1 + 0^2 x_2 &= 0 \\ x_0 + 1x_1 + 1^2 x_2 &= 0 \\ x_0 + 2x_1 + 2^2 x_2 &= 0 \end{aligned}$$

This only has the trivial solution $x_0 = x_1 = x_2 = 0$. So $1, t, t^2$ are linearly independent. □

(7) Which of the following are bases of $\mathbb{R}^3$? Why or why not?

$$A = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}), B = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\4\end{bmatrix}), C = (\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix})$$

Solution:
A is not a basis because 2 vectors can at most span a plane but not all of $\mathbb{R}^3$.
To check whether B is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix}1 & 2 & 0\\2 & 3 & -1\\0 & 4 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\0 & -1 & -1\\0 & 0 & 0\end{bmatrix}$$

Since we have a 0-row, the vectors in B do not span $\mathbb{R}^3$. Hence B is not a basis.
To check whether C is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix}1 & 2 & 0\\2 & 3 & 1\\0 & 4 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0\\0 & -1 & 1\\0 & 0 & 8\end{bmatrix}$$

The echelon form has no 0-row. So C spans $\mathbb{R}^3$. Further we see from the echelon form that C is linearly independent. So C is a basis. □