Math 2135 - Assignment 6

Due October 11, 2024

(1) Compute the inverse if possible:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -3 & 2 & 4 \\ 0 & 1 & -2 \\ 1 & -3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

Solution: Since A is not square, it does not have an inverse.

Row reduce $[B, I_3]$:

$$\begin{bmatrix} -3 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & -3 & 4 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ -3 & 2 & 4 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 4 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -7 & 16 & 1 & 0 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 0 & 3 & 1 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 10 & 4 \\ 0 & 1 & 0 & 1 & 8 & 3 \\ 0 & 0 & 1 & 1/2 & 7/2 & 3/2 \end{bmatrix}$$

So

$$B^{-1} = \begin{bmatrix} 1 & 10 & 4 \\ 1 & 8 & 3 \\ 1/2 & 7/2 & 3/2 \end{bmatrix}.$$

For C^{-1} find the reduced echelon form of $[C, I_3]$:

$$\begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Since the echelon form of C has a zero row, C is not invertible. □

(2) Let $A, B \in \mathbb{R}^{n \times n}$ be invertible. Show $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$.

Solution: Multiplication yields $AB \cdot B^{-1}A^{-1} = A \cdot I_n \cdot A^{-1} = I_n$. Hence $B^{-1} \cdot A^{-1}$ is the inverse of AB. □

(3) A matrix $C \in \mathbb{R}^{n \times m}$ is called a **left inverse** of a matrix $A \in \mathbb{R}^{m \times n}$ if $CA = I_n$ (the $n \times n$ identity matrix).

(a) Show that if A has a left inverse C, then $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$.

(b) Give an example of a matrix A that has a left inverse but is not invertible.

Solution:

(a) Multiply $Ax = b$ by C on the left to get $Cb = CAx = I_n x = x$. Hence $x = Cb$ is the unique solution of $Ax = b$.

(b) E.g. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has the left inverse $C = [1\ 0]$ since $CA = [1]$. Still A is not invertible because there is no right inverse B such that $AB = I_2$ (alternatively because A is not square).

□

(4) Prove that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad - bc = 0$.

Hint: Show that the columns of A are linearly dependent. Consider the cases $a = 0$ and $a \neq 0$ separately.

Solution: Assume $ad - bc = 0$.

Case, $a = 0$: Then $bc = 0$ yields $b = 0$ or $c = 0$. Hence

$$A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix} \text{ or } A = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}.$$

Either way, the columns of A are linearly dependent.

Case, $a \neq 0$: Then $d = \frac{bc}{a}$. Hence

$$A = \begin{bmatrix} a & b \\ c & \frac{bc}{a} \end{bmatrix}$$

and the second column is $\frac{b}{a}$ times the first column. Hence the columns of A are linearly dependent.

By the Inverse Matrix Theorem, a matrix with linearly dependent columns is not invertible. □

(5) Let A be an **upper triangular matrix**, that is,

$$A = \begin{bmatrix} a_{11} & \dots & \dots & a_{1n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_{nn} \end{bmatrix}$$

with zeros below the diagonal. Show

(a) A is invertible iff there are no zeros in the diagonal of A.

(b) If A^{-1} exists, it is an upper triangular matrix as well.
Hint: When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, what happens to the n columns on the right?

Solution:

(a) By the Invertible Matrix Theorem A is invertible iff the columns of A are linearly independent.
If the triangular matrix A has no zero diagonal entries, then A is actually in echelon form and its columns are linearly independent (hence A is invertible).
Conversely if a diagonal entry of A is 0, then there is no pivot in this column of the echelon form of A. Hence the columns of A are not linearly independent (and A not invertible).

(b) When row reducing $[A, I_n]$ to $[I_n, A^{-1}]$, we only need to obtain ones in the diagonal of A (by scaling rows) and zeros above the diagonal of A (by adding multiples of one row to rows above). These operations transform I_n into an upper triangular matrix A^{-1}.

□

(6) Assume that $f\colon \mathbb{R}^n \to \mathbb{R}^n,\ x \mapsto A \cdot x$, is bijective. Show that $A \in \mathbb{R}^{n\times n}$ is invertible.

Give a formula for the inverse function f^{-1}.

Hint: Use that f is surjective and the Invertible Matrix Theorem.

Solution: Since f is onto, $\operatorname{Col} A = F^n$. Since the n columns of A span F^n, they form a basis of F^n by the Basis Theorem. But if the columns of A form a basis, then A is invertible by the Invertible Matrix Theorem.

$f^{-1}\colon F^n \to F^n,\ x \mapsto A^{-1} \cdot x$, which can be verified by composing $x \to A^{-1}x$ with $f\colon x \to Ax$ and observing that one gets the identity function on F^n. □

(7) (a) What is the inverse of the rotation R by angle α counter clockwise around the origin in $\mathbb{R}^2$? What is the standard matrix of R^{-1}?

(b) What is the inverse of a reflection S on a line through the origin in $\mathbb{R}^2$? What can you say about the standard matrix B of S and its inverse? You do not have to write down B for this.

Solution:

(a) R^{-1} is just the rotation by α clockwise (or by $-\alpha$ counter clockwise). R has standard matrix

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

and R^{-1} has standard matrix

$$A^{-1} = \frac{1}{\cos\alpha^2 + \sin\alpha^2} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix}$$

(b) Reflecting twice puts every point x back to itself. Hence any reflection is its own inverse, $S^{-1} = S$. the standard matrix B of S also satisfies $B^{-1} = B$.

□

(8) True of false? Explain your answer.

(a) If A, B are square matrices with $AB = I_n$, then A and B are invertible.

(b) If A is invertible, then A^T is invertible.

(c) Let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent. Then $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto Ax$ is not injective.

Solution:

(a) True. By the Invertible Matrix Theorem $A^{-1} = B$ and $B^{-1} = A$.

(b) True. Recall that $(AB)^T = B^T A^T$. Hence $(A^{-1})^T$ is the inverse of A^T.

(c) True. If $Ax = b$ is inconsistent, then A does not have a pivot in every row. Since A is square, this means that it does not have a pivot in every column either. So $x \mapsto Ax$ is not injective.

□