Math 2135 - Assignment 5

Due October 4, 2024

Problems 1-5 are review material for the first midterm on October 2. Solve them before Wednesday!

(1) Let

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 \\ 1 & 4 & 0 & 7 \\ 2 & -1 & -3 & 8 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 5 \\ -8 \end{bmatrix}$$

(a) Give the solution for $Ax = b$ in parametrized vector form.
(b) Give vectors that span the null space of A.

Solution: (a) Row reduce the augmented matrix

$$[A\, b] = \begin{bmatrix} 0 & 3 & 1 & 2 & 6 \\ 1 & 4 & 0 & 7 & 5 \\ 2 & -1 & -3 & 8 & -8 \end{bmatrix} \text{ (flip rows 1 and 2 to eliminate in first column)}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 7 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 2 & -1 & -3 & 8 & -8 \end{bmatrix} \text{ (add (-2)*row 1 to row 3)}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 7 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 0 & -9 & -3 & -6 & -18 \end{bmatrix} \text{ (add 3*row 2 to row 3)}$$

$$\to \begin{bmatrix} 1 & 4 & 0 & 7 & 5 \\ 0 & 3 & 1 & 2 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now $x_4 = t$ and $x_3 = s$ for $s, t \in \mathbb{R}$ are free. Next

$$3x_2 + s + 2t = 6 \text{ yields } x_2 = 2 - \frac{1}{3}s - \frac{2}{3}t$$

Finally

$$x_1 + 4(2 - \frac{1}{3}s - \frac{2}{3}t) + 0s + 7t = 5 \text{ yields } x_1 = -3 + \frac{4}{3}s - \frac{13}{3}t$$

Separating the solution into the constant part, multiples of s and of t yields the parametrized vector form

$$x = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -13/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$

(b) Note that $p = (-3, 2, 0, 0)^T$ above is a particular solution of $Ax = b$ and that

$$s \begin{bmatrix} 4/3 \\ -1/3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -13/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} \text{ for } s, t \in \mathbb{R}$$

is the set of solutions of $Ax = 0$, i.e. the null space of A. Hence Nul $A =$ Span$\{(4/3, -1/3, 1, 0)^T,\ (-13/3, -2/3, 0, 1)^T\}$. □

(2) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\-1\\1\end{bmatrix} \text{ and } T(\begin{bmatrix}3\\4\end{bmatrix}) = \begin{bmatrix}0\\1\\-2\end{bmatrix}.$$

What is the standard matrix of T?

Solution: Solve 2 linear systems to see how to write the unit vectors as linear combinations of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\4\end{bmatrix}$,

$$e_1 = (-2)\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}3\\4\end{bmatrix} \qquad e_2 = \frac{3}{2}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{2}\begin{bmatrix}3\\4\end{bmatrix}.$$

Next use the linearity of T to get

$$T(e_1) = (-2)T(\begin{bmatrix}1\\2\end{bmatrix}) + 1T(\begin{bmatrix}3\\4\end{bmatrix}) = (-2)\begin{bmatrix}2\\-1\\1\end{bmatrix} + 1\begin{bmatrix}0\\1\\-2\end{bmatrix} = \begin{bmatrix}-4\\3\\-4\end{bmatrix}$$

$$T(e_2) = \frac{3}{2}\begin{bmatrix}2\\-1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}0\\1\\-2\end{bmatrix} = \begin{bmatrix}3\\-2\\5/2\end{bmatrix}.$$

Now the standard matrix of T is just

$$A = [T(e_1)\ T(e_2)] = \begin{bmatrix}-4 & 3\\3 & -2\\-4 & 5/2\end{bmatrix}$$

□

(3) Let $T\colon \mathbb{R}^n \to \mathbb{R}^n, x \mapsto Ax$, be a surjective linear map. Show that T is injective as well.

Solution: By a Theorem of class, if T is surjective, then A must have a pivot in every row. Since A is square, it then also has a pivot in every column. But that means that the columns of A are linearly independent and that T is injective by the same Theorem. □

(4) True or false? Explain your answer.

(a) If $Ax = b$ is inconsistent for some vector b, then A cannot have a pivot in every column.

(b) If vectors $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent and $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$, then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is linear independent.

(c) The range of $T\colon \mathbb{R}^n \to \mathbb{R}^m, x \mapsto Ax$, is the span of the columns of A.

Solution:

(a) False. If $Ax = b$ is inconsistent for some b, then the echelon form of A must have a zero row. So A cannot have a pivot in the last row. But it can still have pivots in every column if there are more rows than columns.

(b) True. By a Theorem from class, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent iff one of the vectors is in the span of the previous vectors. Now assume $\mathbf{v}_1, \mathbf{v}_2$ are linearly independent. Then $\mathbf{v}_1$ is not 0 and $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$. Further assume $\mathbf{v}_3$ is not in the span of $\mathbf{v}_1, \mathbf{v}_2$. Then no $\mathbf{v}_i$ is in the span of $\mathbf{v}_1, \dots, \mathbf{v}_{i-1}$. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

(c) True. Ax is a linear combination of the columns of A, and $T(\mathbb{R}^n)$ is just the set of all these linear combinations, i.e., the span.

□

(5) (a) Give examples of square matrices A, B such that neither A nor B is 0 (the matrix with all entries 0) but $AB = 0$.
(b) If the first two columns of a matrix B are equal, what can you say about the columns of AB?
(c) We can view vectors in $\mathbb{R}^n$ as $n \times 1$ matrices. For $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$ compute $\mathbf{u}^T \cdot \mathbf{v}$ and $\mathbf{u} \cdot \mathbf{v}^T$. Interpret the results.

Solution:
(a) E.g. $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Note AB is 0 but BA is not.
(b) If $\mathbf{b}_i$ is the i-th column of B, then $A\mathbf{b}_i$ is the i-th column of AB. So if the first two columns of a matrix B are equal, then the first two columns of AB are equal as well.
(c) $\mathbf{u}^T \cdot \mathbf{v}$ is a 1×1-matrix but really just the dot-product of the vectors $\mathbf{u}$ and $\mathbf{v}$.
$\mathbf{u} \cdot \mathbf{v}^T$ is the 3×3-matrix with single entries of $\mathbf{u}$ and $\mathbf{v}$ multiplied.

□

(6) Prove for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$ that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Solution: Multiplying $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ and cancelling $ad - bc$ yields the identity matrix. Hence the given matrix is the inverse of A.

□

(7) Are the following invertible? Give the inverse if possible.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

Solution:

$$A^{-1} = \frac{1}{2(-9) - 1 \cdot 4} \begin{bmatrix} -9 & -1 \\ -4 & 2 \end{bmatrix}, \quad B^{-1} \text{ does not exist since } 2(-6) - (-3)4 = 0$$

Since C has a zero column, for every matrix D the product DC has a zero column as well. So DC can never be the identity matrix. Thus C is not invertible.

□

(8) A **diagonal matrix** A has all entries 0 except on the diagonal, that is,

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Under which conditions is A invertible and what is A^{-1}?

Solution: We see that

$$A^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 & \dots & 0 \\ 0 & a_{22}^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}^{-1} \end{bmatrix}$$

is the only choice for the inverse of A, and it exists iff all diagonal entries $a_{11}, \dots, a_{nn}$ are distinct from 0. $\square$