Math 2135 - Assignment 4

Due September 27, 2024

(1) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix},\ T(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

(a) Use linearity to find $T(e_1)$ and $T(e_2)$ for the unit vectors e_1, e_2 in $\mathbb{R}^2$.

(b) Give the standard matrix for T and determine $T(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

Solution:

(a) First write the unit vectors as linear combinations of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\2\end{bmatrix}$.

Solve

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

to get $x = -\frac{1}{2}$ and $y = \frac{1}{2}$. By the linearity of T we obtain

$$\begin{aligned}
T(\begin{bmatrix}1\\0\end{bmatrix}) &= T(-\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}3\\2\end{bmatrix}) \\
&= -\frac{1}{2}T(\begin{bmatrix}1\\2\end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix}3\\2\end{bmatrix}) \\
&= -\frac{1}{2}\begin{bmatrix}2\\0\\-3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-2\\2\\1\end{bmatrix} \\
&= \begin{bmatrix}-2\\1\\2\end{bmatrix}
\end{aligned}$$

Similarly we compute that

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{3}{4}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{4}\begin{bmatrix}3\\2\end{bmatrix}$$

and hence obtain

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{3}{4}\begin{bmatrix}2\\0\\-3\end{bmatrix} - \frac{1}{4}\begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{bmatrix}2\\-1/2\\-5/2\end{bmatrix}$$

(b) By (a) we know the standard matrix of T is

$$A = \begin{bmatrix}-2 & 2\\1 & -1/2\\2 & -5/2\end{bmatrix}.$$

Thus $T(\begin{bmatrix}x\\y\end{bmatrix}) = A \cdot \begin{bmatrix}x\\y\end{bmatrix}$.

$\square$

(2) Is the following injective, surjective, bijective? What is its range?

$$T : \mathbb{R}^3 \to \mathbb{R}^2,\ x \mapsto \begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot x$$

Solution: Not injective because x_1 is free in $A \cdot x = \mathbf{0}$. Alternatively, the columns of A are linearly dependent. So T is not injective (Theorem 12 of Section 1.9).

Surjective because A is in row echelon form and has no 0-rows (Theorem 12 of Section 1.9). Hence its range is just its codomain $\mathbb{R}^2$.

Bijective means injective and surjective. Hence T is not bijective because it is not injective. □

(3) Is the following injective, surjective, bijective?

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto \begin{bmatrix} 1 & -1 & 2 \\ -2 & 0 & 1 \\ 3 & -1 & 1 \end{bmatrix} \cdot x$$

Solution: Row reduce the standard matrix of T to get

$$\begin{array}{cccc} 1 & -1 & 2 & \\ -2 & 0 & 1 & \\ 3 & -1 & 1 & \\ \hline 1 & -1 & 2 & \\ 0 & -2 & 5 & 2I + II \\ 0 & 2 & -5 & -3I + III \\ \hline 1 & -1 & 2 & \\ 0 & -2 & 5 & \\ 0 & 0 & 0 & -II + III \end{array}$$

T is not injective because not every column of the echelon form of A has a pivot. In particular x_3 is free in $A \cdot x = \mathbf{0}$.

T is not surjective because the echelon form of A has a zero row. Hence $Ax = y$ is not consistent for every $y \in \mathbb{R}^3$.

Since T is neither injective nor surjective, it is certainly not bijective. □

(4) True or False? Explain why and correct the false statements to make them true.

(a) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.

(b) Not every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(x) = Ax$ for some matrix A.

(c) The composition of any two linear transformations is linear as well.

Solution:

(a) True because every vector is a linear combination of unit vectors.

(b) False. Every linear $T : \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(x) = Ax$ for its standard matrix A.

(c) True. If S, T are linear, then

$$S(T(u + v)) = S(T(u) + T(v)) = S(T(u) + S(T(v))$$

for all u, v in the domain of T and

$$S(T(cu)) = S(cT(u)) = cS(T(u))$$

for $c \in \mathbb{R}$. Hence S composed with T is linear.

□

(5) True or False? Explain why and correct the false statements to make them true.

(a) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $x \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.
(b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $x \in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.
(c) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one.
(d) There is a surjective linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}^4$.

Solution:

(a) False. Any function $T : \mathbb{R}^n \to \mathbb{R}^m$ maps every vector $x \in \mathbb{R}^n$ onto some vector $T(x)$ in $\mathbb{R}^m$.
The correct statement is: $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if for every vector $y \in \mathbb{R}^m$ there is some vector $x \in \mathbb{R}^n$ such that $T(x) = y$.
(b) False. Any function $T : \mathbb{R}^n \to \mathbb{R}^m$ maps every vector $x \in \mathbb{R}^n$ onto the unique vector $T(x)$ in $\mathbb{R}^m$.
The correct statement is: $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if any 2 distinct vectors $x_1, x_2 \in \mathbb{R}^n$ are mapped to distinct vectors $T(x_1), T(x_2)$.
(c) True. If A is the 2×3 standard matrix of T, then solving $A \cdot x = \mathbf{0}$ will always yield at least one free variable.
(d) False. If A is the 3×4 standard matrix of T, then $Ax = y$ cannot have a solution for every $y \in \mathbb{R}^4$ since the echelon form of A has at least one zero row.

$\square$

(6) If defined, compute the following for the matrices

$$A = \begin{bmatrix} 2 & 1 & -4 \\ 3 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 \\ -3 & 4 \\ 2 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}$$

Else explain why the computation is not defined.

(a) AB (b) BA (c) AC (d) $A + C$ (e) $AB + 2C$

Solution:

$$AB = \begin{bmatrix} 2 \cdot 1 + 1 \cdot (-3) - 4 \cdot 2 & 2 \cdot (-1) + 1 \cdot 4 - 4 \cdot 0 \\ 3 \cdot 1 - 1 \cdot (-3) + 1 \cdot 2 & 3 \cdot (-1) - 1 \cdot 4 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} -9 & 2 \\ 8 & -7 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 \cdot 2 - 1 \cdot 3 & 1 \cdot 1 - 1 \cdot (-1) & 1 \cdot (-4) - 1 \cdot 1 \\ * & * & * \\ * & * & * \end{bmatrix} = \begin{bmatrix} -1 & 2 & -5 \\ 6 & -7 & 16 \\ 4 & 2 & -8 \end{bmatrix}$$

AC is undefined since the length of A's rows and the length of C's columns are not the same.
$A + C$ is undefined since the sizes of A and C don't match.

$$AB + 2C = \begin{bmatrix} -9 & 2 \\ 8 & -7 \end{bmatrix} + \begin{bmatrix} -4 & 2 \\ 4 & -6 \end{bmatrix} = \begin{bmatrix} -13 & 4 \\ 12 & -13 \end{bmatrix}$$

$\square$

(7) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ first rotate points around the origin by $60°$ counter clockwise and then reflect points at the line with equation $y = x$. Give the standard matrix for T.

(a) Recall the standard matrix A for the rotation R by $60°$ from class.
(b) Determine the standard matrix B for the reflection S at the line with equation $y = x$ (a sketch will help).
(c) Since T is the composition of S and R, compute the standard matrix C of T as the product of B and A. Careful about the order!

Solution:

(a) The standard matrix for the rotation R by $\alpha = 60°$ is

$$A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

(b) The reflection S at the diagonal flips the unit vectors, i.e., $T(e_1) = e_2$ and $T(e_2) = e_1$. Hence the standard matrix of S is

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(c) Since $T = S \circ R$ (S after R), its standard matrix is

$$C = BA = \begin{bmatrix} \sin\alpha & \cos\alpha \\ \cos\alpha & -\sin\alpha \end{bmatrix}$$

□

(8) Continuation of (7): What is the standard matrix for $U \colon \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points at the line with equation $y = x$ and then rotates points around the origin by 60° counter clockwise? Compare T and U.

Solution: Since $U = R \circ S$ (R after R), its standard matrix is

$$AB = \begin{bmatrix} -\sin\alpha & \cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix}$$

Comparing the result with the standard matrix of $S \circ R$, we see that the result is not the same. Order of function composition and matrix multiplication matters!

□