Math 2135 - Assignment 4

Due September 27, 2024

(1) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix},\ T(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

(a) Use linearity to find $T(e_1)$ and $T(e_2)$ for the unit vectors e_1, e_2 in $\mathbb{R}^2$.

(b) Give the standard matrix for T and determine $T(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

(2) Is the following injective, surjective, bijective? What is its range?

$$T : \mathbb{R}^3 \to \mathbb{R}^2,\ x \mapsto \begin{bmatrix}0 & 2 & -1\\0 & 0 & 3\end{bmatrix} \cdot x$$

(3) Is the following injective, surjective, bijective?

$$T : \mathbb{R}^3 \to \mathbb{R}^3,\ x \mapsto \begin{bmatrix}1 & -1 & 2\\-2 & 0 & 1\\3 & -1 & 1\end{bmatrix} \cdot x$$

(4) True or False? Explain why and correct the false statements to make them true.

(a) A linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by the images of the unit vectors in $\mathbb{R}^n$.

(b) Not every linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ can be written as $T(x) = Ax$ for some matrix A.

(c) The composition of any two linear transformations is linear as well.

(5) True or False? Explain why and correct the false statements to make them true.

(a) $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto $\mathbb{R}^m$ if every vector $x \in \mathbb{R}^n$ is mapped onto some vector in $\mathbb{R}^m$.

(b) $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if every vector $x \in \mathbb{R}^n$ is mapped onto a unique vector in $\mathbb{R}^m$.

(c) A linear map $T : \mathbb{R}^3 \to \mathbb{R}^2$ cannot be one-to-one.

(d) There is a surjective linear transformation $T\colon \mathbb{R}^3 \to \mathbb{R}^4$.

(6) If defined, compute the following for the matrices

$$A = \begin{bmatrix}2 & 1 & -4\\3 & -1 & 1\end{bmatrix},\ B = \begin{bmatrix}1 & -1\\-3 & 4\\2 & 0\end{bmatrix},\ C = \begin{bmatrix}-2 & 1\\2 & -3\end{bmatrix}$$

Else explain why the computation is not defined.

(a) AB (b) BA (c) AC (d) $A + C$ (e) $AB + 2C$

(7) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ first rotate points around the origin by $60°$ counter clockwise and then reflect points at the line with equation $y = x$. Give the standard matrix for T.

(a) Recall the standard matrix A for the rotation R by $60°$ from class.

(b) Determine the standard matrix B for the reflection S at the line with equation $y = x$ (a sketch will help).

(c) Since T is the composition of S and R, compute the standard matrix C of T as the product of B and A. Careful about the order!

(8) Continuation of (7): What is the standard matrix for $U\colon \mathbb{R}^2 \to \mathbb{R}^2$ which first reflects points at the line with equation $y = x$ and then rotates points around the origin by $60°$ counter clockwise? Compare T and U.