Math 2135 - Assignment 3

Due September 20, 2024

(1) Which of the following sets of vectors are linearly independent?

(a) $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 0 \end{bmatrix}$

Solution:

Check whether $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}$ has a non-trivial solution:

(a) Row reduce

$$\begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 7 & -2 \end{bmatrix} \sim \dots$$

Note this is the coefficient matrix of the linear system, not the augmented matrix but there the last column is all 0s anyway.
3 pivots, no free variables for the null space, columns are linearly independent.

(b)

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & -11 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -14 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots, 1 free variable for the null space, columns are linearly dependent.

□

(2) Explain whether the following are true or false:

(a) Vectors $\mathbf{v}_1, \mathbf{v}_2, v_3$ are linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3$.
(b) A subset $\{\mathbf{v}\}$ containing just a single vector is linearly dependent iff $\mathbf{v} = \mathbf{0}$.
(c) Two vectors are linearly dependent iff they lie on a line through the origin.
(d) There exist four vectors in $\mathbb{R}^3$ that are linearly independent.

Solution:

(a) **True.** If $\mathbf{v}_2 = c_1\mathbf{v}_1 + c_3\mathbf{v}_3$ for some $c_1, c_3 \in \mathbb{R}$, then $c_1\mathbf{v}_1 + (-1)\mathbf{v}_2 + c_3\mathbf{v}_3$ is a non-trivial linear combination that yields the zero vector $\mathbf{0}$.

(b) **True.** Assume $\mathbf{v} = \mathbf{0}$ is the zero vector. Then $1\mathbf{v} = \mathbf{0}$ is a non-trivial linear combination that yields the zero vector. Hence $\mathbf{v}$ is linearly dependent.
Conversely, assume $\mathbf{v} \neq \mathbf{0}$. Then $c\mathbf{v} = \mathbf{0}$ only if $c = 0$. Hence $\mathbf{v}$ is linearly independent.

(c) **True.** Assume $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent. Then either $\mathbf{v}_1 = \mathbf{0}$ or $\mathbf{v}_2$ is a multiple of $\mathbf{v}_1$ by a Theorem from class. In either case $\mathbf{v}_1, \mathbf{v}_2$ lie in a line through the origin.
Conversely, assume $\mathbf{v}_1, \mathbf{v}_2$ lie in a line through the origin. Then one is a multiple of the other and both are linearly dependent.

(d) **False.** For $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4 \in \mathbb{R}^3$ solving $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = 0$ yields a coefficient matrix with 3 rows and 4 columns. Since there are only 3 rows, there can be at most 3 pivots. Hence there is at least 1 free variable in the solution of the system. Hence $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are linearly dependent.

□

(3) Show: If any of the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ is the zero vector (say $\mathbf{v}_i = \mathbf{0}$ for $i \leq n$), then $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent.
Solution:
Assume $\mathbf{v}_i = \mathbf{0}$. Then

$$0\mathbf{v}_1 + \cdots + 0\mathbf{v}_{i-1} + 1\mathbf{v}_i + 0\mathbf{v}_{i+1} + \cdots + 0\mathbf{v}_n = \mathbf{0}$$

is a non-trivial linear combination that yields the zero vector. Hence $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly dependent. □

(4) Show: If $n > m$, then any n vectors $\mathbf{a}_1, \dots, \mathbf{a}_n \in \mathbb{R}^m$ are linearly dependent.
Solution:
Compare with (2)(d). Solving $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots x_n\mathbf{a}_n = \mathbf{0}$ yields a coefficient matrix with m rows and n columns. There can be at most m pivots. Hence there are at least $n - m > 0$ free variables in the solution of the system. Hence $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly dependent. □

(5) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.
(a) $f : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+3 \end{bmatrix}$
(b) $g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$
(c) $h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x|+|y| \\ 2x \end{bmatrix}$
Solution:
For example
(a) $f(0 \cdot \mathbf{0}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \neq \mathbf{0} = 0 \cdot f(\mathbf{0})$
(b) $g(2 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \cdot g(\begin{bmatrix} 1 \\ 1 \end{bmatrix})$
(c) $h((-1) \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \neq \begin{bmatrix} -1 \\ -2 \end{bmatrix} = (-1) \cdot h(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$

□

(6) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \; T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Use the linearity of T to compute $T(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix})$. What is the issue with the latter?

Solution:

$$T(\begin{bmatrix}2\\3\\0\end{bmatrix}) = T(2\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\cdot\begin{bmatrix}0\\1\\0\end{bmatrix}) = 2\cdot T(\begin{bmatrix}1\\0\\0\end{bmatrix}) + 3\cdot T(\begin{bmatrix}0\\1\\0\end{bmatrix}) = 2\cdot\begin{bmatrix}-1\\2\\0\end{bmatrix} + 3\cdot\begin{bmatrix}-3\\0\\1\end{bmatrix} = \begin{bmatrix}-11\\4\\3\end{bmatrix}$$

$T(\begin{bmatrix}1\\2\\3\end{bmatrix})$ cannot be computed with the given information since $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is not a linear combination of $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$. □

(7) Let $T\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix}, \; T(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}.$$

(a) Use the linearity of T to find $T(\begin{bmatrix}1\\0\end{bmatrix})$ and $T(\begin{bmatrix}0\\1\end{bmatrix})$.

(b) Determine $T(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

Solution:

(a) First write the unit vectors as linear combinations of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}3\\2\end{bmatrix}$.

Solve

$$x\begin{bmatrix}1\\2\end{bmatrix} + y\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

to get $x = -\frac{1}{2}$ and $y = \frac{1}{2}$. By the linearity of T we obtain

$$\begin{aligned} T(\begin{bmatrix}1\\0\end{bmatrix}) &= T(-\frac{1}{2}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{2}\begin{bmatrix}3\\2\end{bmatrix}) \\ &= -\frac{1}{2}T(\begin{bmatrix}1\\2\end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix}3\\2\end{bmatrix}) \\ &= -\frac{1}{2}\begin{bmatrix}2\\0\\-3\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-2\\2\\1\end{bmatrix} \\ &= \begin{bmatrix}-2\\1\\2\end{bmatrix} \end{aligned}$$

Similarly we compute that

$$\begin{bmatrix}0\\1\end{bmatrix} = \frac{3}{4}\begin{bmatrix}1\\2\end{bmatrix} - \frac{1}{4}\begin{bmatrix}3\\2\end{bmatrix}$$

and hence obtain

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{3}{4}\begin{bmatrix}2\\0\\-3\end{bmatrix} - \frac{1}{4}\begin{bmatrix}-2\\2\\1\end{bmatrix} = \begin{bmatrix}2\\-1/2\\-5/2\end{bmatrix}$$

(b) By (a) we know the standard matrix of T is

$$A = \begin{bmatrix}-2 & 2\\1 & -1/2\\2 & -5/2\end{bmatrix}.$$

Thus $T(\begin{bmatrix}x\\y\end{bmatrix}) = A \cdot \begin{bmatrix}x\\y\end{bmatrix}$.

□

(8) Give the standard matrices for the following linear transformations:

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$, $\begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}2x+y\\x\\-x+y\end{bmatrix}$;

Solution:

Just take the coefficient matrix of the transformation to get its standard matrix

$$A = \begin{bmatrix}2 & 1\\1 & 0\\-1 & 1\end{bmatrix}$$

□

(b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.

Solution:

The function is $S : \mathbb{R}^2 \to \mathbb{R}^2$, $\begin{bmatrix}x\\y\end{bmatrix} \mapsto \frac{1}{2}\begin{bmatrix}x\\y\end{bmatrix}$ and has standard matrix $A = \begin{bmatrix}\frac{1}{2} & 0\\0 & \frac{1}{2}\end{bmatrix}$.

□