Math 2135 - Assignment 3

Due September 20, 2024

(1) Which of the following sets of vectors are linearly independent?

(a) $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 0 \end{bmatrix}$

(2) Explain whether the following are true or false:
   (a) Vectors $\mathbf{v}_1, \mathbf{v}_2, v_3$ are linearly dependent if $\mathbf{v}_2$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_3$.
   (b) A subset $\{\mathbf{v}\}$ containing just a single vector is linearly dependent iff $\mathbf{v} = \mathbf{0}$.
   (c) Two vectors are linearly dependent iff they lie on a line through the origin.
   (d) There exist four vectors in $\mathbb{R}^3$ that are linearly independent.

(3) Show: If any of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is the zero vector (say $\mathbf{v}_i = \mathbf{0}$ for $i \leq n$), then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly dependent.

(4) Show: If $n > m$, then any n vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ are linearly dependent.

(5) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.
   (a) $f : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+1 \\ y+3 \end{bmatrix}$
   (b) $g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$
   (c) $h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix}$

(6) Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \ T(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

Use the linearity of T to compute $T(\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix})$. What is the issue with the latter?

(7) Let $T \colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$T(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, \ T(\begin{bmatrix} 3 \\ 2 \end{bmatrix}) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$

   (a) Use the linearity of T to find $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$.
   (b) Determine $T(\begin{bmatrix} x \\ y \end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

(8) Give the standard matrices for the following linear transformations:
   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x+y \\ x \\ -x+y \end{bmatrix}$;
   (b) the function S on $\mathbb{R}^2$ that scales all vectors to half their length.