Math 2135 - Assignment 2

Due September 14, 2024

(1) Is $\mathbf{b}$ a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2$?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

Solution:
No, $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution for x_1, x_2. □

(2) Is $\mathbf{b} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ for

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}?$$

Solution:

Row reduce the augmented matrix for the system $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

$$\begin{array}{rrrrl} 1 & -2 & -6 & 11 & \\ 0 & 3 & 7 & -5 & \\ 1 & -2 & 5 & 9 & \\ \hline 1 & -2 & -6 & 11 & \\ 0 & 3 & 7 & -5 & \\ 0 & 0 & 11 & 20 & -I + II \end{array}$$

So a solution exists and $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3$. □

(3) For which values of a is $\mathbf{b}$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} a \\ -3 \\ -5 \end{bmatrix}$$

Solution:
Row reduce the augmented matrix

$$\begin{array}{rrcl} 1 & -2 & a & \\ 0 & 1 & -3 & \\ -2 & 7 & -5 & \\ \hline 1 & -2 & a & \\ 0 & 1 & -3 & \\ 0 & 3 & -5 + 2a & 2I + III \\ \hline 1 & -2 & a & \\ 0 & 1 & -3 & \\ 0 & 0 & 4 + 2a & -3II + III \end{array}$$

Solvable for $a = -2$. Hence $\mathbf{y}$ is in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$ iff $a = -2$. $\square$

(4) Find vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^3$ that span the plane in $\mathbb{R}^3$ with equation $x - 2y + 3z = 0$. How many do you need?

Hint: Write down a parametrized solution for the equation.

Solution:

The solution is $z = t, y = s, x = 2s - 3t$, hence

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Ever point on the plane is a linear combination of $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. $\square$

(5) Are the following true or false? Explain your answers.

(a) For every $A \in \mathbb{R}^{2\times 3}$ with 2 pivots, $Ax = 0$ has a nontrivial solution.

Solution:

True: $Ax = 0$ is consistend and has exactly $3 - 2$ free variables, hence a non-trivial solution. $\square$

(b) For every $A \in \mathbb{R}^{2\times 3}$ with 2 pivots and every $\mathbf{b} \in \mathbb{R}^2$, $Ax = \mathbf{b}$ is consistent.

Solution:

True: The echelon form of $Ax = b$ has 2 non-zero rows. $\square$

(c) The vector $3\mathbf{v}_1$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2$.

Solution:

True: $3\mathbf{v}_1 = 3\mathbf{v}_1 + 0\mathbf{v}_2$. $\square$

(d) For $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$, $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2)$ is always a plane through the origin.

Solution:

False: It might be a plane or a line or just the origin (if $\mathbf{v_1} = \mathbf{v_2} = 0$). E.g., $\mathrm{Span}\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}\}$ is just the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ because the second vector is a multiple of the first. $\square$

(6) [1, cf. Section 1.5, Ex 17] Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$. Express both solution sets in parametric vector form. Give a geometric description of the solution sets.

Solution:
We solve $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

The solution set is a line through the point $(8, -4, 0)$ spanned by the vector $(-1, -1, 1)$. For the homogeneous system $A\mathbf{x} = \mathbf{0}$ we obtain

$$\mathbf{x} = r \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad r \in \mathbb{R}.$$

This solution set is a line through the origin spanned by the vector $(-1, -1, 1)$. □

(7) (a) Which of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the nullspace of A, Null A?

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

Solution:
$\mathbf{u}, \mathbf{v} \in \text{Null}\, A$ since $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$; $\mathbf{w} \notin \text{Null}\, A$. □

(b) Solve $A\mathbf{x} = \mathbf{0}$ and give the solution in parametric vector form.
Solution:
We solve $A\mathbf{x} = \mathbf{0}$ and reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 4 & 0 \\ 2 & -4 & 1 & 0 & 0 \\ -3 & 6 & 2 & 7 & 0 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccc|c} 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The variables x_2 and x_4 are free. We obtain

$$\begin{aligned} x_1 &= 2r + s \\ x_2 &= r \\ x_3 &= -2s \\ x_4 &= s. \end{aligned}$$

The solution in parametric vector form is

$$\mathbf{x} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad r, s \in \mathbb{R}.$$

□

(c) Find vectors $v_1, \ldots, v_k \in \mathbb{R}^4$ such that $\operatorname{Null} A = \operatorname{Span}\{v_1, \ldots, v_k\}$.
Solution:

$$\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

□

(8) Show the following:

Theorem. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$. Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{p} + \operatorname{Null} A = \{\mathbf{p} + \mathbf{v} \mid \mathbf{v} \in \operatorname{Null} A\}.$$

Hint: For the proof suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$ and use 2 steps:
(a) Show that if $\mathbf{v}$ is in $\operatorname{Null} A$, then $\mathbf{p} + \mathbf{v}$ is also a solution for $A\mathbf{x} = \mathbf{b}$.
(b) Show that if $\mathbf{q}$ is a solution for $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} - \mathbf{p}$ is in $\operatorname{Null} A$.
Solution:

(a) Assume $\mathbf{v}$ is in the null space of A. We know that $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{p} = \mathbf{b}$. Thus, using that matrix multiplication distributes over a sum of vectors by a theorem from class,

$$A(\mathbf{p} + \mathbf{v}) = A\mathbf{p} + A\mathbf{v} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Hence $\mathbf{p} + \mathbf{v}$ is a solution of $A\mathbf{x} = \mathbf{b}$.

(b) Assume $\mathbf{q}$ is a solution of $A\mathbf{x} = \mathbf{b}$. Then $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{q} = \mathbf{b}$. Thus

$$A(\mathbf{q} - \mathbf{p}) = A\mathbf{q} - A\mathbf{p} = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Hence $\mathbf{q} - \mathbf{p}$ is in $\operatorname{Null} A$. Thus $\mathbf{q} = \mathbf{p} + (\mathbf{q} - \mathbf{p})$ is in $\mathbf{p} + \operatorname{Null} A$.

We have show that every solution of $Ax = \mathbf{b}$ is in $\mathbf{p} + \operatorname{Null} A$ and conversely.

□

References

[1] David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Addison-Wesley, 5th edition, 2015.