Math 2135 - Assignment 2

Due September 14, 2024

(1) Is $\mathbf{b}$ a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2$?

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

(2) Is $\mathbf{b} \in \operatorname{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ for

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}?$$

(3) For which values of a is $\mathbf{b}$ in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$?

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} a \\ -3 \\ -5 \end{bmatrix}$$

(4) Find vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^3$ that span the plane in $\mathbb{R}^3$ with equation $x - 2y + 3z = 0$. How many do you need?
Hint: Write down a parametrized solution for the equation.

(5) Are the following true or false? Explain your answers.
  (a) For every $A \in \mathbb{R}^{2\times 3}$ with 2 pivots, $Ax = 0$ has a nontrivial solution.
  (b) For every $A \in \mathbb{R}^{2\times 3}$ with 2 pivots and every $\mathbf{b} \in \mathbb{R}^2$, $Ax = \mathbf{b}$ is consistent.
  (c) The vector $3\mathbf{v}_1$ is a linear combination of the vectors $\mathbf{v}_1, \mathbf{v}_2$.
  (d) For $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^3$, $\operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2)$ is always a plane through the origin.

(6) [1, cf. Section 1.5, Ex 17] Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$. Express both solution sets in parametric vector form. Give a geometric description of the solution sets.

(7) (a) Which of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the nullspace of A, $\operatorname{Null} A$?

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

  (b) Solve $A\mathbf{x} = \mathbf{0}$ and give the solution in parametric vector form.
  (c) Find vectors $v_1, \dots, v_k \in \mathbb{R}^4$ such that $\operatorname{Null} A = \operatorname{Span}\{v_1, \dots, v_k\}$.

(8) Show the following:

Theorem. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$. Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{p} + \operatorname{Null} A = \{\mathbf{p} + \mathbf{v} \mid \mathbf{v} \in \operatorname{Null} A\}.$$

Hint: For the proof suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$ and use 2 steps:
(a) Show that if $\mathbf{v}$ is in $\operatorname{Null} A$, then $\mathbf{p} + \mathbf{v}$ is also a solution for $A\mathbf{x} = \mathbf{b}$.
(b) Show that if $\mathbf{q}$ is a solution for $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} - \mathbf{p}$ is in $\operatorname{Null} A$.

References

[1] David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Addison-Wesley, 5th edition, 2015.