Math 2135 - Assignment 1

Due September 6, 2024

Solve all systems of linear equations by row reduction (Gaussian elimination) and give the solutions in parametric vector form.

(1) Do the following 4 planes intersect in a point? Which?

$$\begin{aligned} x + 5y + 3z &= 16 \\ 2x + 10y + 8z &= 34 \\ 4x + 20y + 15z &= 67 \\ x + 6y + 5z &= 21 \end{aligned}$$

Solution:

Row reduce the augmented matrix:

1	5	3	16	I
2	10	8	34	II
4	20	15	67	III
1	6	5	21	IV
1	5	3	16	I
0	0	2	2	$-2 \cdot I + II$
0	0	3	3	$-4 \cdot I + III$
0	1	2	5	$-I + IV$
1	5	3	16	
0	1	2	5	old 4th row
0	0	2	2	
0	0	3	3	
1	5	3	16	
0	1	2	5	
0	0	2	2	
0	0	0	0	$-3/2 \cdot III + IV$

Solve backwards to get the intersection point $x_3 = 1, x_2 = 3, x_1 = -2$. □

(2) Add an equation of a line to the equation

$$2x + 3y = 4$$

such that the resulting system has (a) no solution, (b) exactly one solution, (c) infinitely many solutions.

Solution:

(a) parallel line $2x + 3y = 5$
(b) e.g. $x = 0$
(c) same line, e.g., $-2x - 3y = -4$

$\square$

(3) Solve the system of linear equations with augmented matrix

$$\begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

Solution:

Row reduce the augmented matrix. First we flip first and second row.

$$\begin{array}{ccccl} 2 & -4 & 1 & 0 & I \\ 0 & 0 & 2 & 4 & II \\ -3 & 6 & 2 & 7 & III \\ \hline 2 & -4 & 1 & 0 & I \\ 0 & 0 & 1 & 2 & 1/2 \cdot II \\ 0 & 0 & \frac{7}{2} & 7 & 3/2 \cdot I + III \\ \hline 2 & -4 & 1 & 0 & \\ 0 & 0 & 1 & 2 & \\ 0 & 0 & 0 & 0 & -7/2 \cdot II + III \end{array}$$

Solve backwards $x_3 = 2$, $x_2 = t$ for $t \in \mathbb{R}$, $x_1 = -1 + 2t$. $\square$

(4) Solve the system of linear equations with augmented matrix

$$\begin{bmatrix} 2 & 2 & 1 & 8 & 2 \\ 2 & 0 & 0 & 8 & 2 \\ 2 & 6 & 3 & 8 & 1 \end{bmatrix}$$

Solution:

Row reduce the augmented matrix to get

$$\begin{array}{ccccc} 2 & 2 & 1 & 8 & 2 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 4 & 2 & 0 & -1 \\ \hline 2 & 2 & 1 & 8 & 2 \\ 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}$$

Since the last equation is $0 = -1$, there is no solution. $\square$

(5) Let $A \in \mathbb{R}^{3 \times 2}$. Explain why $Ax = b$ cannot have a solution x for all $b \in \mathbb{R}^3$.
Solution:
A has 3 rows, 2 columns. After row reduction there can be at most 2 pivots (2 non-zero rows). Hence a row echelon form of

A must have at least 1 zero row. For some $b \in \mathbb{R}^3$ the echelon form of the augmented matrix $(A|b)$ must then have a row $(0, 0, e)$ for $e \neq 0$. Hence $Ax = b$ is inconsistent by a Theorem from class. □

(6) [1, cf. Section 1.5, Ex 17] Let

$$A = \begin{bmatrix} 2 & 2 & 4 \\ -4 & -4 & -8 \\ 0 & -3 & -3 \end{bmatrix}, \quad b = \begin{bmatrix} 8 \\ -16 \\ 12 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve $Ax = b$ and $Ax = \mathbf{0}$. Express both solution sets in parametric vector form. Give a geometric description of the solution sets.

Solution:

Row reduce the augmented matrix to get

$$\begin{array}{cccc} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \\ \hline 2 & 2 & 4 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \\ \hline 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}$$

Solution for the homogenous system $Ax = 0$: a line through the origin

$$x = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

Solution for the inhomogenous system $Ax = b$: a shifted line parallel to the previous one

$$x = \begin{bmatrix} 8 \\ -4 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, t \in \mathbb{R}$$

□

(7) [1, cf. Section 1.5, Ex 11] Let

$$A = \begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the equations $Ax = b$ and $Ax = \mathbf{0}$. Express both solution sets in parametric vector form.

Solution:
Solution for $Ax = b$:

$$x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}, r, s, t \in \mathbb{R}$$

□

(8) Are the following true or false? Explain your answers.

(a) Any system of linear equations with strictly less equations than variables has infinitely many solutions.
Solution:
False: There might be no solution at all, like for $x+y+z = 0$ and $0 = 1$. □

(b) Different sequences of elementary row reductions can transform one matrix to distinct matrices in row echelon form.
Solution:
True: E.g., the matrix $A = [2\ 4]$ can be reduced to echelon form $[2\ 4]$ or $[1\ 2]$. □

(c) A consistent system has exactly one solution.
Solution:
False: There might free variables and hence infinitely many solutions, like for $x + y = 0$. □

References

[1] David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Addison-Wesley, 5th edition, 2015.