Math 2001 - Assignment 11

Due April 13, 2018

(1) Your country issues stamps for 3 cents and for 7 cents. Show that every amount of at least 12 cents can be combined from them, that is, for every $n \in \mathbb{N}, n \geq 12$, there exist $a, b \in \mathbb{N}_0$ such that

$$n = 3a + 7b.$$

Use strong induction on n with base cases $n = 12, 13, 14$.

(2) Let $p_1, p_2, \dots$ denote the list of all primes. Show that for integers $a = \Pi_{i\in\mathbb{N}}\, p_i^{e_i}, b = \Pi_{i\in\mathbb{N}}\, p_i^{f_i}$ with $e_i, f_i \in \mathbb{N}_0$ for $i \in \mathbb{N}$,

$$\operatorname{lcm}(a, b) = \Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i, f_i)}.$$

(3) Show for all $a, b \in \mathbb{N}$:

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$$

Hint: Use the formula for gcd and lcm from class and the previous problem.

(4) Prove or disprove that the following relations are reflexive, symmetric, antisymmetric, transitive:
   (a) $<$ on $\mathbb{Z}$
   (b) $\neq$ on $\mathbb{Z}$

(5) Prove or disprove that the following relations are reflexive, symmetric, antisymmetric, transitive:
   (a) $S = \{(1, 2), (1, 3), (2, 2), (2, 3), (3, 2)\}$ on $\{1, 2, 3\}$
   (b) $\subseteq$ on the power set $P(A)$ of a set A

(6) Prove or disprove that the following relations are reflexive, symmetric, antisymmetric, transitive:
   (a) $|$ (divides) on $\mathbb{N}$
   (b) $R = \{(x, y) \in \mathbb{R} \colon |x - y| \leq 1\}$