QUANTIFIED STATEMENTS

PETER MAYR (MATH 2001, CU BOULDER)

$\forall$ ("for all") is called the **universal quantifier**.
$\exists$ ("there exists") is called the **existential quantifier**.

Let A be a set and $P(x)$ a statement about elements $x \in A$.

- $\forall x \in A : P(x)$ — True if $P(x)$ is true for all $x \in A$.
- $\exists x \in A : P(x)$ — True if $P(x)$ is true for some $x \in A$.

Negations

- $\sim \big(\forall x \in A : P(x)\big)$
- $\sim \big(\exists x \in A : P(x)\big)$

Examples. Use quantifiers and sets to express the following statements. Are they true? Give their negation.

(1) For all $x \in \mathbb{R}$, x^2 is greater or equal to 0.

(2) $\sin x \in [-1, 1]$ for any $x \in \mathbb{R}$.

(3) Every integer is even.

(4) There exists $x \in \mathbb{Z}$ such that $x^2 = 3$.

(5) $2^x \geq 1$ for some $x \in \mathbb{R}$.

(6) Some subsets of $\mathbb{N}$ are finite.

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Multiple quantifiers

(1) $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}$: $x < y$

(2) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}$: $x < y$

(3) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}$: $x < y$

(4) $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}$: $x < y$

Note. Order of $\forall$ and $\exists$ matters!

Truth of quantified statement as game

$\forall x \in A\ \exists y \in B :\ P(x, y)$

Examples. Are the following statements true? Give their negation.

(1) For every $x \in \mathbb{R}$ there is some $y \in \mathbb{N}$ such that $x \leq y$.

(2) There exists some $y \in \mathbb{N}$ such that $x \leq y$ for every $x \in \mathbb{R}$.

(3) For every $a, b \in \mathbb{R}$ with $a < b$ there exists $x \in \mathbb{Q}$ such that $x \in [a, b]$.

(4) Every polynomial has a root in $\mathbb{R}$.