Math 2001 - Practice Midterm 1

(1) Use the Axiom of Replacement or Specification to describe:
(a) $C = \{1, 9, 25, 49, 81\}$
(b) $D =$ the set of subsets of $\mathbb{Z}$ of even size

Solution
(a) Replacement $C = \{(2n+1)^2 \ : \ n \in \{0, 1, 2, 3, 4\}\}$
Specification $C = \{n \in \{1, \dots, 81\} \ : \ n \text{ is an odd square } \}$
(b) Specification $D = \{X \in P(\mathbb{Z}) \ : \ |X| \text{ is even } \}$

(2) Prove without Venn diagrams that for all sets A, B in a universe U:

$$\overline{A} - \overline{B} = B - A$$

Solution: First show $\overline{A} - \overline{B} \subseteq B - A$: Let $x \in \overline{A} - \overline{B}$. Then $x \in \overline{A}$ and $x \notin \overline{B}$. That means $x \notin A$ and $x \in B$. Hence $x \in B - A$.

For the converse $B - A \subseteq \overline{A} - \overline{B}$ just do the steps above backwards: Let $x \in B - A$.

...

Then $x \in \overline{A} - \overline{B}$.

This shows that the two sets are equal.

(3) Let $P(A)$ denote the power set of A. Is the following true for all sets A, B?

$$P(A \times B) \subseteq P(A) \times P(B)$$

Prove it or give a counter-example.

Solution: $A \times B$ is the set of all pairs (a, b) where $a \in A, b \in B$. The elements of $P(A \times B)$ are arbitrary sets of pairs (a, b) where $a \in A, b \in B$.

On the other hand, the elements of $P(A) \times P(B)$ are pairs of subsets of A and subsets of B. Since sets of pairs and pairs of sets are not the same, elements of $P(A \times B)$ are not in $P(A) \times P(B)$.

For an explicit counter-example consider e.g. $A = \{a\}, B = \{b\}$. Then $A \times B = \{(a, b)\}$ and

$$P(A \times B) = \{\emptyset, \{(a, b)\}\}.$$

On the other hand $P(A) = \{\emptyset, A\}$ and $P(B) = \{\emptyset, B\}$. Hence

$$P(A) \times P(B) = \{(\emptyset, \emptyset), (\emptyset, B), (A, \emptyset), (A, B)\}.$$

Clearly $\emptyset \in P(A\times B)$ but $\emptyset \notin P(A)\times P(B)$. Hence $P(A\times B) \not\subseteq P(A) \times P(B)$.

(4) Write using quantifiers and logical operations:
  (a) The square of any real number is non-negative.
  (b) There exists an integer x such that $x^y = x$ for all integers y.

Solution
  (a) $\forall x \in \mathbb{R} : \ x^2 \geq 0$
  (b) $\exists x \in \mathbb{Z} \ \forall y \in \mathbb{Z} : \ x^y = x$

(5) Which of the following are true? Explain why or why not.
  (a) $\forall x \in \mathbb{Z} \ \forall y \in \mathbb{Z} : xy = y$
  (b) $\forall x \in \mathbb{Z} \ \exists y \in \mathbb{Z} : xy = y$
  (c) $\exists x \in \mathbb{Z} \ \forall y \in \mathbb{Z} : xy = y$
  (d) $\exists x \in \mathbb{Z} \ \exists y \in \mathbb{Z} : xy = y$

Solution
  (a) False, e.g., for $x = 2, y = 1$.
  (b) True, for any x pick $y = 0$ to get $xy = y$.
  (c) True, pick $x = 1$ to get for any y that $xy = y$.
  (d) True, e.g. for $x = y = 0$.

(6) Negate without using the phrase "It is not true that..." and without "$\sim$":
  (a) $\forall n \in \mathbb{N} \ \exists A \in P(\mathbb{N}) : |A| > n$
  (b) $\forall$ polynomial p $\exists x \in \mathbb{R} : \ p(x) = 0$ or p is constant
  (c) $x + y = 0$ and $x - y = 0$ iff $x = 0$ and $y = 0$.

Solution
  (a) $\exists n \in \mathbb{N} \ \forall A \in P(\mathbb{N}) : |A| \leq n$
  (b) $\exists$ polynomial p $\forall x \in \mathbb{R} : \ p(x) \neq 0$ and p is not constant
  (c) $x + y = 0$ and $x - y = 0$ iff $x \neq 0$ or $y \neq 0$.