Math 2001 - Assignment 13

Due December 5, 2025

(1) (a) Give domain, codomain, and range of $f\colon \mathbb{Z} \to \mathbb{N}$, $x \mapsto x^2 + 1$. What is $f(3)$?
(b) Is f one-to-one, onto, bijective?
(c) Determine $f(\{2x \ : \ x \in \mathbb{Z}\})$ and $f^{-1}(\{1, 2, 3, \ldots, 10\})$.

Solution.

(a) domain $\mathbb{Z}$, codomain $\mathbb{N}$, range $\{x^2+1 \ : \ x \in \mathbb{Z}\}$, $f(3) = 10$
(b) not injective since e.g. $f(1) = f(-1)$,
not surjective since e.g. $\nexists x \in \mathbb{Z} : f(x) = 3$,
hence not bijective
(c) $f(\{2x \ : \ x \in \mathbb{Z}\}) = \{4x^2 + 1 \ : \ x \in \mathbb{Z}\}$,
$f^{-1}(\{1, 2, 3, \ldots, 10\}) = \{-3, -2, -1, 0, 1, 2, 3\}$

(2) Give examples for
(a) a function $f\colon \mathbb{N} \to \mathbb{N}$ that is not injective but surjective;
(b) a function $g\colon \{1, 2, 3\} \to \{1, 2\}$ that is neither injective nor surjective;
(c) a bijective function $h\colon \{1, 2, 3\} \to \{1, 2\}$.

Solution.

(a) E.g. $f(x) = \lceil \frac{x}{2} \rceil$, the smallest integer greater or equal to $\frac{x}{2}$
(b) any constant function
(c) Not possible: Because the codomain is smaller than the domain, there is no injective h.

(3) Let $f\colon A \to B, g\colon B \to C$. Show that
(a) If $g \circ f$ is injective, then f is injective.
(b) If $g \circ f$ is surjective, then g is surjective.

Hint: Use contrapositive proofs.

Give examples for f, g on $A = B = C = \mathbb{N}$ such that
(c) $g \circ f$ is injective but g is not injective;
(d) $g \circ f$ is surjective but f is not surjective.

Proof.

(a) Assume f is not injective, that is, we have $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$ as well. Hence $g \circ f$ is not injective.
(b) Assume g is not surjective, that is, $g(B) \neq C$. Since $g(f(A)) \subseteq g(B)$, $g(f(A))$ cannot be all of C either. Hence $g \circ f$ is not surjective.
(c) If $g \circ f$ is injective, then g restricted to $f(A)$ has to be injective. But it does not matter what g does on $B - f(A)$.

E.g., let $f\colon \mathbb{N} \to \mathbb{N},\ x \mapsto 2x,\ g\colon \mathbb{N} \to \mathbb{N},\ x \mapsto \lceil \frac{x}{2} \rceil$ where $\lceil r \rceil$ is the smallest integer z such that $z \geq r$. Then $g \circ f = \mathrm{id}_{\mathbb{N}}$ is injective but g is not.

(d) If $g \circ f$ is surjective, then $g(f(A)) = C$ but it does mean that $f(A)$ needs to be all of B.
E.g. as in (c) $g \circ f = \mathrm{id}_{\mathbb{N}}$ is surjective but f is not.

(4) (a) Show that

$$f : \mathbb{R} - \{1\} \to \mathbb{R} - \{2\}, x \mapsto \frac{2x+1}{x-1}$$

is bijective.

(b) Determine f^{-1}.

Solution. (a) Injective: Let $x, y \in \mathbb{R} - \{1\}$ such that $f(x) = f(y)$. Show $x = y$. We have

$$\begin{aligned} \frac{2x+1}{x-1} &= \frac{2y+1}{y-1} \\ (2x+1)(y-1) &= (2y+1)(x-1) \\ \ldots &= \ldots \\ x &= y \end{aligned}$$

Hence f is injective.

Surjective: Let $y \in \mathbb{R} - \{2\}$ such that $f(x) = y$. Solve for $x \in \mathbb{R} - \{1\}$.

$$\begin{aligned} y &= \frac{2x+1}{x-1} \\ y(x-1) &= 2x+1 \\ -y-1 &= x(-y+2) \\ \frac{y+1}{y-2} &= x \end{aligned}$$

So we found $x \in \mathbb{R} - \{1\}$ such that $f(x) = y$ and hence f is surjective.

Thus f is bijective.

(b) From the proof of surjectivity, we see

$$f^{-1}\colon \mathbb{R} - \{2\} \to \mathbb{R} - \{1\},\ y \mapsto \frac{y+1}{y-2}$$

Note. Checking surjectivity and finding the inverse is pretty much the same work. So you may just try to find f^{-1} straight away without bothering about injectivity and surjectivity first.

If $f(x) = y$ does not have a unique solution, then you'll see a failure of injectivity or surjectivity anyway.

(5) Try to you find an inverse for $f\colon \mathbb{R} \to \mathbb{R}^+,\ x \mapsto e^{x^3+1}$. Is f bijective?

Solution: Given $y \in \mathbb{R}^+$, find $x \in \mathbb{R}$ such that $f(x) = y$. So we solve

$$\begin{aligned} e^{x^3+1} &= y \\ x^3 + 1 &= \log y \\ x &= (\log y - 1)^{\frac{1}{3}} \end{aligned}$$

So

$$f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}, y \mapsto (\log y - 1)^{\frac{1}{3}}$$

By checking $f \circ f^{-1} = \mathrm{id}_{\mathbb{R}^+}$ and $f^{-1} \circ f = \mathrm{id}_{\mathbb{R}}$ we see that f is bijective.

(6) Let U be a set, and let c be the function on the power set of U that maps every set to its complement in U, i.e.,

$$c\colon P(U) \to P(U), X \mapsto \bar{X}.$$

Determine c^{-1} if it exists.

Solution: Given $Y \in P(U)$, find $X \in P(U)$ such that $\bar{X} = Y$. Take the complement again to get $X = \bar{\bar{X}} = \bar{Y}$.

Hence $c = c^{-1}$ is its own inverse. This can also be seen by $c \circ c = \mathrm{id}_{P(U)}$.