Math 2001 - Assignment 11

Due November 14, 2025

(1) Let $p_1, p_2, \dots$ denote the list of all primes. Show that for integers $a = \Pi_{i\in\mathbb{N}}\, p_i^{e_i}, b = \Pi_{i\in\mathbb{N}}\, p_i^{f_i}$ with $e_i, f_i \in \mathbb{N}_0$ for $i \in \mathbb{N}$,

$$\operatorname{lcm}(a,b) = \Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)}.$$

Proof:
a) First note that $\Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)}$ is an integer multiple of a since

$$\Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)} = a \cdot \underbrace{\Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)-e_i}}_{\in\mathbb{Z}}.$$

Similar $\Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)}$ is an integer multiple of b.
b) Next let $m = \Pi_{i\in\mathbb{N}}\, p_i^{g_i)}$ for $g_i \in \mathbb{N}_0$ a common multiple of a and b. Let $i \in \mathbb{N}$. Then $p_i^{e_i}$ divides m and by the Fundamental Theorem of Arithmetic, $p_i^{e_i}$ divides $p_i^{g_i}$. Hence $e_i \leq g_i$. Similarly $f_i \leq g_i$. Together they imply that $\max(e_i, f_i) \leq g_i$.

Thus for any common multiple m of a and b we have $\Pi_{i\in\mathbb{N}}\, p_i^{\max(e_i,f_i)} \leq m$ and the former is $\operatorname{lcm}(a,b)$. □

(2) Show for all $a, b \in \mathbb{N}$:

$$\gcd(a,b) \cdot \operatorname{lcm}(a,b) = ab$$

Hint: Use the formula for gcd and lcm from class and the previous problem.

Proof: Let $a = \prod_{i\in\mathbb{N}} p_i^{e_i}, b = \prod_{i\in\mathbb{N}} p_i^{f_i}$. By a lemma from class

$$\gcd(a,b) = \prod_{i\in\mathbb{N}} p_i^{\min(e_i,f_i)} \qquad \operatorname{lcm}(a,b) = \prod_{i\in\mathbb{N}} p_i^{\max(e_i,f_i)}$$

Now $ab = \prod_{i\in\mathbb{N}} p_i^{e_i+f_i}$ and

$$\gcd(a,b)\operatorname{lcm}(a,b) = \prod_{i\in\mathbb{N}} p_i^{\min(e_i,f_i)+\max(e_i,f_i)}.$$

Both numbers on the right hand side are equal since for any $e, f \in \mathbb{N}_0$

$$e + f = \min(e,f) + \max(e,f).$$

The proof of that is by case distinction:
Case 1, $e \leq f$: Then $\min(e,f) = e, \max(e,f) = f$ and $\min(e,f) + \max(e,f) = e + f$.
Case 2, $e > f$: Similar. □

Prove or disprove that the following relations are reflexive, symmetric, antisymmetric, transitive. Which are equivalences, which partial orders?

(3) $\neq$ on $\mathbb{Z}$

Solution.
- not reflexive since e.g. it is not true that $0 \neq 0$
- symmetric since $\forall x, y \in \mathbb{Z}\colon x \neq y \Rightarrow y \neq x$
- not antisymmetric since e.g. $0 \neq 1$ and $1 \neq 0$ but it is not true that $0 = 1$
- not transitive since e.g. $0 \neq 1$ and $1 \neq 0$ but it is not true that $0 \neq 0$
- Hence $\neq$ is neither an equivalence nor a partial order.

(4) $\subseteq$ on the power set $P(A)$ of a set A

Solution.
- reflexive since $X \subseteq X$ for every set $X \in P(A)$.
- not symmetric if $A \neq \emptyset$. Then $\emptyset \subseteq A$ but $A \not\subseteq \emptyset$.
- antisymmetric since $X \subseteq Y$ and $Y \subseteq X$ implies $X = Y$ for all $X, Y \in P(A)$.
- transitive since $X \subseteq Y$ and $Y \subseteq Z$ implies $X \subseteq Z$ for all $X, Y, Z \in P(A)$.
- Hence $\subseteq$ is not an equivalence but a partial order.

(5) $|$ (divides) on $\mathbb{N}$

Solution.
- reflexive since $x|x$ for every $x \in \mathbb{N}$.
- not symmetric since e.g. $1|2$ but $2 \not| 1$.
- antisymmetric since $x|y$ and $y|x$ implies $x = y$ for all $x, y \in \mathbb{N}$.
- transitive since $x|y$ and $y|z$ implies $x|z$ for all $x, y, z \in \mathbb{N}$.
- Hence $|$ is not an equivalence but a partial order.

(6) $R = \{(x, y) \in \mathbb{R}\colon |x - y| \leq 1\}$

Solution.
- reflexive since $|x - x| = 0 \leq 1$ for every $x \in \mathbb{R}$.
- symmetric since $|x - y| = |y - x|$ for all $x, y \in \mathbb{R}$.
- not antisymmetric since e.g. $|0 - 1| \leq 1$ and $|1 - 0| \leq 1$ but $0 \neq 1$
- not transitive since e.g. $|0 - 1| \leq 1$ and $|1 - 2| \leq 1$ but $|0 - 2| > 1$
- Hence R is neither an equivalence nor a partial order.