Math 2001 - Assignment 10

Due November 7, 2025

(1) Prove by induction that for every $q \in \mathbb{R}$ with $q \neq 1$ and for every $n \in \mathbb{N}_0$:

$$1 + q^1 + q^2 + \cdots + q^n = \frac{1 - q^{n+1}}{1 - q}$$

Proof by induction on n:
Induction basis for $n = 0$: $q^0 = 1$ holds.

Induction assumption: Assume the formula holds for a particular $k \in \mathbb{N}$.

Induction step: Show the formula holds for $n = k + 1$.

$$\begin{aligned}
\sum_{i=0}^{k+1} q^i &= \sum_{i=0}^{k} q^i + q^{k+1} \\
&= \frac{1 - q^{k+1}}{1 - q} + q^{k+1} \text{ by induction assumption} \\
&= \frac{1 - q^{k+1}}{1 - q} + \frac{q^{k+1} - q^{k+2}}{1 - q} \\
&= \frac{1 - q^{k+2}}{1 - q}
\end{aligned}$$

Hence the induction step is proved and the formula holds for all $n \in \mathbb{N}$. □

(2) [1, Chapter 10, exercise 2] Show by induction that for every $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof by induction on n:
Induction basis for $n = 1$: $1^2 = 1$ holds.

Induction assumption: Assume the formula holds for a particular $n \in \mathbb{N}$.

Induction step: Show the formula holds for $n + 1$.

$$\sum_{k=1}^{n+1} k^2 = \sum_{k=1}^{n} k^2 + (n+1)^2$$
$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \text{ by induction assumption}$$
$$= \frac{(n+1)[n(2n+1)+6(n+1)]}{6}$$
$$= \frac{(n+1)[2n^2+7n+6)]}{6}$$
$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

Hence the induction step is proved and the formula holds for all $n \in \mathbb{N}$. □

(3) [1, Chapter 10, exercise 8] Show that for every $n \in \mathbb{N}$:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

Proof by induction on n:

Inductive base: For n=1, it is true that $\frac{1}{2!} = 1 - \frac{1}{2!}$.

Induction assumption: For a fixed n we have

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$$

Inductive step: Show

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+2)!}$$

Just start with the left hand side and simplify it using the induction assumption:

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n+1}{(n+2)!} = 1 - \frac{1}{(n+1)!} + \frac{n+1}{(n+2)!} \text{ by induction assumption}$$
$$= 1 - \frac{n+2}{(n+2)!} + \frac{n+1}{(n+2)!}$$
$$= 1 - \frac{1}{(n+2)!}$$

Thus the statement is true for all $n \in \mathbb{N}$. □

(4) Show by induction that for every natural number $n \geq 4$:

$$2^n \geq n^2$$

Proof by induction on n:

Induction basis for $n = 4$: $2^4 \geq 4^2$ holds.

Induction assumption: Assume $2^k \geq k^2$ holds for a particular $k \geq 4$.

Induction step: Show $2^{k+1} \geq k + 1^2$.

Note

$$2^{k+1} = 2 \cdot 2^k \geq 2k^2 \text{ by induction assumption}$$

So we want to still show that

$$2k^2 \geq (k+1)^2. \tag{1}$$

To this end, look at the difference of both sides

$$\begin{aligned} 2k^2 - (k+1)^2 &= k^2 - 2k - 1 \\ &= (k-1)^2 - 2 \\ &\geq 3^2 - 2 \text{ because } k \geq 4. \end{aligned}$$

In particular $2k^2 - (k+1)^2 \geq 0$ which proves (1) and the induction step. □

(5) Prove the following generalization of Euclid's Lemma by induction:

Let $n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{Z}$, p prime. If $p|a_1 \ldots a_n$, then $p|a_i$ for some $i \leq n$.

Proof:

- **Basis step,** $n = 1$**:** $p|a_1$ and the statement is true.
- **Induction hypothesis:** For a fixed $k \in \mathbb{N}$, if $p|a_1 \cdots a_k$, then $p|a_i$ for some $i \in \{1, \ldots, k\}$.
- **Induction step:** Show the statement follows for $n = k+1$. Assume $p|\underbrace{a_1 \cdots a_k}_{=b} a_{k+1}$.

  By Euclid's Lemma, $p|b$ or $p|a_{k+1}$.
  - If $p|b$, then $p|a_i$ for some $i \in \{1, \ldots, k\}$ by induction assumption.
  - Else $p|a_{k+1}$.

  In any case $p|a_1$ or $p|a_2$ or $\ldots p|a_k$ or $p|a_{k+1}$. The induction step is proved and so is the Lemma. □

(6) Define a sequence of integers $a_1 := 1, a_2 := 1$ and

$$a_n := 2a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$

Prove that a_n is odd for all $n \in \mathbb{N}$ by strong induction.

Proof by strong induction on n: Induction basis for $n = 1, 2$: holds by definition of a_1, a_2.

Strong induction assumption: Assume a_i is odd for all $i \leq k$.

Induction step: Show that a_{k+1} is odd.

By definition $a_{k+1} = 2a_k + a_{k-1}$. Since a_{k-1} is odd by the strong induction assumption, a_{k+1} is the sum of an even and an odd integer. Hence a_{k+1} is odd. □

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 3rd edition, 2018. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`