Math 2001 - Assignment 6

Due October 10, 2025

(1) How many standard Colorado license plates (3 numbers followed by 3 letters) have at least one number or letter repeated?

Solution: number of all plates as computed in class: $10^3 26^3$

number of plates without repetition of any number or letter: $10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24$

Subtract the above numbers to get the number of plates with one number or letter repeated: $10^3 26^3 - 10 \cdot 9 \cdot 8 \cdot 26 \cdot 25 \cdot 24 =$ 6.344.000

(2) (a) How many different 5-card hands form a Full House? Pick 5 cards from a standard 52-card deck such that 3 are of a kind (same value) and the remaining 2 are of a kind.

(b) How many different 5-card hands contain 4-of-a-kind (same value)?

(c) Is Full House of 4-of-a-kind stronger in Poker?

Solution.

(a) For a Full House, pick 1 kind out of 13 values and choose 3 cards of the 4 with that value. Moreover pick 1 kind out of the remaining 12 values and choose 2 cards of the 4 with that value.

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6 = 3744$$

(b) For 4-of-a-kind pick 1 kind out of 13 values and all 4 cards of that value. Then pick one more card from the remaining 48.

$$\binom{13}{1}\binom{48}{1} = 13 \cdot 48 = 624.$$

(c) Since there are 6 times more Full Houses than 4-of-a-kinds, the latter is stronger.

(3) The street map of Manhattan is a grid with avenues running North-South and streets East-West. How many different ways are there from the corner of 2nd Ave and C Street to the corner of 6th Ave and F Street if you only ever go South and West (never North or East).

Solution: You need to go 4 blocks West and 3 blocks South. Every single path is described by a list of instructions, e.g: On the first corner W, on the second W, then S,...

Hence every path corresponds to a list of length 7 with 4 Ws and 3 Ss in it. How many such lists are there? There are 3 position to choose S out of 7, hence

$$\binom{7}{3}$$

(4) In a freshman class of 300 students 145 take English, 155 take Calculus and 120 Discrete Math. 90 take English and Calculus, 80 English and Discrete Math, 75 Calculus and Discrete Math, 60 take all three.

How many students take none of these three classes?

Solution. Use inclusion-exclusion to get the number of students that take non of the 3 classes

300	all students
$-(145+155+120)$	students taking E, C, or D
$+(90+80+75)$	students taking E & C, E & D, or C & D
-60	students taking all 3 classes

(5) How many positive integers less or equal 100 are not multiples of 2 or 3 or 5?

Solution
Number of multiples of 2: $\lfloor \frac{100}{2} \rfloor = 50$
Number of multiples of 3: $\lfloor \frac{100}{3} \rfloor = 33$
Number of multiples of 5: $\lfloor \frac{100}{3} \rfloor = 20$
...
Number of integers that are not multiples of 2 or 3 or 5 by inclusion-exclusion:
100 - |multiples of 2| -|multiples of 3|-|multiples of 5| + |multiples of 2 and 3| + |multiples of 2 and 5|+ |multiples of 3 and 5| - |multiples of 2 and 3 and 5|
$= 100 - \lfloor \frac{100}{2} \rfloor - \lfloor \frac{100}{3} \rfloor - \lfloor \frac{100}{5} \rfloor + \lfloor \frac{100}{6} \rfloor + \lfloor \frac{100}{10} \rfloor + \lfloor \frac{100}{15} \rfloor - \lfloor \frac{100}{30} \rfloor$
$= 100 - 50 - 33 - 20 + 16 + 10 + 6 - 3 = 26$

(6) How many permutations of $\{1, 2, 3, 4, 5\}$ have 1 in the first position? How many permutations do not have 1 in first or 2 in second position?

Solution.
Number of permutations with 1 in first position: 4!
Number of permutations with 1 in first and 2 in second: 3!
Number of permutations that do not have 1 in first or 2 in second by inclusion-exclusion: 5!-4!-4!+3! $= 78$.

(7) Use the Binomial Theorem to show for all $n \in \mathbb{N}$:
(a) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$
(b) $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + (-1)^n \binom{n}{n} = 0$
Hint: Which values do you want to assign for x and y in the Binomial Theorem to obtain the expressions above?

Solution.
(a) Compute $(1+1)^n$ in 2 ways:

$$(1+1)^n = 2^n$$

$$(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$$

(b) Compute $(1+(-1))^n$ in 2 ways:

$$(1+(-1))^n = 0^n = 0$$

$$(1+(-1))^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k}(-1)^k = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$