Math 2001 - Assignment 4

Due September 26, 2025

(1) (a) How many different truthtables (Boolean functions) are there for 2 statements x_1, x_2? How many for k statements $x_1, \ldots, x_k$?

(b) Let $f(x_1, x_2, x_3)$ be a Boolean function that is true for the following assignments and false otherwise.

x_1	x_2	x_3	$f(x_1, x_2, x_3)$
T	T	F	T
T	F	T	T
F	T	T	T

Write an expression for $f(x_1, x_2, x_3)$ using only $\wedge, \vee, \sim$.

Solution

(a) A truthtable for 2 statements x_1, x_2 has $2^2 = 4$ rows. For each row we can choose either true or false. Hence we have 2^4 options to make a truthtable.
A truthtable for k statements $x_1, \ldots, x_k$ has 2^k rows. Hence there are 2^{2^k} such tables.

(b) $f(x_1, x_2, x_3)$ and $(x_1 \wedge x_2 \wedge \sim x_3) \vee (x_1 \wedge \sim x_2 \wedge x_3) \vee (\sim x_1 \wedge x_2 \wedge x_3)$ are true at exactly the same assignments. Hence they are equal.

(2) [1, Section 2.7]: Exercises 4,6,7,9,10. Also give the negation of the corresponding statements.

Solution:

4. For all elements X in the powerset of $\mathbb{N}$, we have that X is a subset of $\mathbb{R}$.
Every subset of $\mathbb{N}$ is a subset of $\mathbb{R}$.
True since $\mathbb{N} \subseteq \mathbb{R}$
Negation: $\exists X \in P(\mathbb{N}),\ X \not\subseteq \mathbb{R}$

6. There exists a natural number n such that every subset of $\mathbb{N}$ has less than n elements.
False, $\{1, \ldots, n\}$ is a subset with n elements
Negation: $\forall n \in \mathbb{N} \exists X \in P(\mathbb{N}),\ |X| \geq n$

7. For every subset X of $\mathbb{N}$ there exists an integer n such that X has size n.
False, $X = \mathbb{N}$ is a subset of $\mathbb{N}$ of infinite size.
Negation: $\exists X \subseteq \mathbb{N}\ \forall n \in \mathbb{Z}:\ |X| \neq n$

9. For every integer n there exists an integer m such that $m = n + 5$.
True, $m = n + 5$ is the integer we are looking for.
Negation: $\exists n \in \mathbb{Z} \forall m \in \mathbb{Z}:\ m \neq n + 5$

10. There exists an integer m for every n such that $m = n + 5$.
False, the same number m cannot work for $n = 0$ and $n = 1$.
Negation: $\forall m \in \mathbb{Z} \forall \exists n \in \mathbb{Z}:\ m \neq n + 5$

(3) Formulate the following sentences using quantifiers and logical operations. Are they true? Negate them.

   (a) For all integers n we have that $n(n+1)$ is even.
   Solution $\forall n \in \mathbb{Z}\ n(n+1)$ is even.
   True because one of n or $n+1$ is even.
   Negation: $\exists n \in \mathbb{Z}\ n(n+1)$ is odd.
   (b) There exists a real number z such that $x+z=x$ for every real x.
   Solution $\exists z \in \mathbb{R} \forall x \in \mathbb{R}\ x+z=x$
   True for $z=0$.
   Negation: $\forall z \in \mathbb{R}\ \exists x \in \mathbb{R}\ x+z \neq z$
   (c) Every real number is smaller than some integer.
   Solution $\forall x \in \mathbb{R} \exists z \in \mathbb{Z}\ x<z$
   True.
   Negation: $\exists x \in \mathbb{R} \forall x \in \mathbb{Z}\ x \geq z$

(4) Negate the following sentences. Are they true?
   (a) If x^2 is rational, then so is x.
   Solution $x^2 \in \mathbb{Q} \Rightarrow x \in \mathbb{Q}$.
   False, e.g, $2 \in \mathbb{Q}$ and $\sqrt{2} \notin Q$
   Negation: $x^2 \in \mathbb{Q}$ and $x \notin \mathbb{Q}$.
   (b) $xy=0$ iff $x=0$ or $y=0$
   True, Negation $xy \neq 0$ iff $x=0$ or $y=0$
   $xy=0$ iff $x \neq 0$ and $y \neq 0$
   (c) The derivative of a polynomial function f is 0 iff f is constant.
   True, Negation: The derivative of a polynomial function f is 0 iff f is not constant.
   (d) $\exists x \in \mathbb{R} : x^2=-1$
   False, Negation: $\forall x \in \mathbb{R} : x^2 \neq -1$
   (e) $\forall r \in \mathbb{R} : \sin(r\pi)=0 \Leftrightarrow r$ is an integer
   Negation: $\exists r \in \mathbb{R} : \sin(r\pi)=0$ iff r is not an integer

(5) True or false? Give a proof or a counter-example:
   (a) $\forall x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x+y=1$
   False, counter-example: x=y=0
   (b) $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x+y=1$
   True: for $x \in \mathbb{R}$ we have $y=1-x$ such that $x+y=1$
   (c) $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R}\ x+y=1$
   False: Suppose we have such a fixed $x \in \mathbb{R}$. Then for $y=-x$ we'd have $x+y=0 \neq 1$. Hence there cannot be a single x that makes $x+y=1$ true for all $y \in \mathbb{R}$.
   (d) $\exists x \in \mathbb{R}\ \exists y \in \mathbb{R}\ x+y=1$
   True: e.g. $x=0, y=1$

(6) Write as complete English sentences. True or false? Negate:
   (a) $\forall a \in \mathbb{R}\ \exists b \in \mathbb{R}\ \forall c \in \mathbb{R} :\ a<b \Rightarrow c<b$
   Solution: For all $a \in \mathbb{R}$ there exists $b \in \mathbb{R}$ such that for all $c \in \mathbb{R}$, if $a<b$ then $c<b$.
   True: Let $a \in \mathbb{R}$ arbitrary. Next choose $b \in \mathbb{R}$ such that $a \not< b$, say $b=a$. Then $\forall c \in \mathbb{R} :\ a<b \Rightarrow c<b$ is true. (Note that by the choice of b, the assumption $a<b$ of the implication is false. Hence FALSE $\Rightarrow c<b$ is true.)
   Negation: $\exists a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \exists c \in \mathbb{R} :\ a<b \wedge b \leq c$.

Note that the negation is clearly false. Hence again the original statement must be true!

(b) $\forall$ set A $\forall$ set B $\exists$ set C : $A \cup B = C$.
Solution: For all sets A and B there exists a set C that is the union of A and B.
True by Axiom of Unions in Zermelo-Fraenkel Set Theory.
Negation: $\exists$ set A $\exists$ set B $\forall$ set C : $A \cup B \neq C$.
There exist sets A and B whose union is not a set.

(c) $\forall x, y \in \mathbb{R}\ \forall \varepsilon > 0\ \exists \delta > 0 : |x - y| < \varepsilon \Rightarrow |2x - 2y| < \delta$
Solution: For all reals x, y and every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \varepsilon$ implies $|2x - 2y| < \delta$.
True for $\delta = 2\varepsilon$.
Negation: $\exists x, y \in \mathbb{R}\ \exists \varepsilon > 0\ \forall \delta > 0 : |x - y| < \varepsilon \wedge |2x - 2y| \geq \delta$.

(d) Simplify:
(i) $\bigcup_{i=0}^{4} [i, 2i + 1]$
(ii) $\bigcap_{n \in \mathbb{N}} \{x \in \mathbb{Z} : x \geq n\}$
(iii) $\bigcup_{x \in [0,1]} \{x\} \times [1, 2]$
(iv) $\bigcup_{x \in [0,1]} \{x\} \times [0, x]$

Solution.
(i) $\bigcup_{i=0}^{4} [i, 2i + 1] = [0, 1] \cup [1, 3] \cup \cdots \cup [4, 9] = [0, 9]$
(ii) $\bigcap_{n \in \mathbb{N}} \{x \in \mathbb{Z} : x \geq n\} = \{1, 2, 3, \dots\} \cap \{2, 3, 4, \dots\} \cap \{3, 4, \dots\} \cap \cdots = \emptyset$ since no integer x is greater than every natural number n
(iii) $\bigcup_{x \in [0,1]} \{x\} \times [1, 2] = \{(x, y) : x \in [0, 1], y \in [1, 2]\} = [0, 1] \times [1, 2]$
(iv) $\bigcup_{x \in [0,1]} \{x\} \times [0, x] = \{(x, y) : x \in [0, 1], y \in [0, x]\} = \{(x, y) : 0 \leq y \leq x \leq 1\}$

References

[1] Richard Hammack. The Book of Proof. Creative Commons, 2nd edition, 2013. Available for free: `http://www.people.vcu.edu/~rhammack/BookOfProof/`