Math 2001 - Assignment 2

Due September 12, 2025

(1) For $U := \{x \in \mathbb{Z} : 1 \leq x \leq 8\}$,
$A = \{1, 2, 3, 4, 5\}$,
$B = \{x \in U : x \text{ is even }\}$,
$C = \{x \in U : x \geq 4\}$ compute:
(a) $A \cap \bar{B}$ (b) $A \cup (B \cap C)$ (c) $(A - B) \cup B$

Solution.
$A \cap \bar{B} = \{1, 3, 5\}$
$A \cup (B \cap C) = \{1, 2, 3, 4, 5, 6, 8\}$
$(A - B) \cup B = A \cup B = \{1, 2, 3, 4, 5, 6, 8\}$

(2) Are the following true for all sets A, B in a universe U?
(a) $A - B = B - A$
(b) $A \cup B \subseteq (A \cap \bar{B}) \cup (B \cap \bar{A})$
Consider Venn diagrams first and then either write a proof that the equations hold or give an example where they fail.

Solution.
(a) $A - B = B - A$ is false. One counterexample is $A = \{1\}, B = \emptyset$
(b) $A \cup B \subseteq (A \cap \bar{B}) \cup (B \cap \bar{A})$ is false. One counterexample is $A = B = \{1\}$.

(3) Show that for all sets A, B, C

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

without Venn diagrams.

Recall that we already showed that the lefthand side is contained in the righthand side. So it only remains to write a proof for the converse,

$$(A \cup B) \cap C \supseteq (A \cap C) \cup (B \cap C).$$

Solution.
Let $x \in (A \cap C) \cup (B \cap C)$. By the definition of $\cup$ we have $x \in (A \cap C)$ or $x \in (B \cap C)$ and hence 2 cases to consider:

Case 1, $x \in (A \cap C)$**:** Then $x \in A$ and $x \in C$ by the definition of $\cap$. Since $x \in A$, we also have $x \in A \cup B$. Together with $x \in C$ this implies that $x \in (A \cup B) \cap C$.

Case 2, $x \in (B \cap C)$**:** Similar to case 1.

In either case $x \in (A \cup B) \cap C$. Hence we proved that $(A \cap C) \cup (B \cap C) \subseteq (A \cup B) \cap C$ $\square$

(4) Show for all sets A, B in the universe U:

$$\overline{A \cup B} = \bar{A} \cap \bar{B} \qquad \text{(de Morgan's law)}$$

First use Venn diagrams. Then write down a proof.

Solution.
Proof of $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$: Let $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, which yields that $x \notin A$ and $x \notin B$. Hence $x \in \bar{A}$ and $x \in \cap\bar{B}$. Thus $x \in \bar{A} \cap \bar{B}$. $\square$

Proof of $\overline{A \cup B} \supseteq \bar{A} \cap \bar{B}$: Let $x \in \bar{A} \cap \bar{B}$. Then $x \in \bar{A}$ and $x \in \bar{B}$. Equivalently $x \notin A$ and $x \notin B$. But then $x \notin A \cup B$, which yields $x \in \overline{A \cup B}$. $\square$

(5) Show that for all sets A, B in the universe U:

$$\overline{A} - \overline{B} = B - A$$

First consider Venn diagrams. Then write down the proof.

Solution.
Show $\bar{A} - \bar{B} \subseteq B - A$: Let $x \in \bar{A} - \bar{B}$. By the definition of $-$, we have $x \in \bar{A}$ and $x \notin \bar{B}$. By the definition of complement this means $x \notin A$ and $x \in B$. Hence $x \in B - A$.

Show $\bar{A} - \bar{B} \subseteq B - A$: Just reorder the previous argument from bottom to top. $\square$

(6) Simplify the following sets and justify your answers:
(a) $\bigcup_{n \in \mathbb{N}} (0, n]$ (b) $\bigcap_{n=1}^{3} \{nz : z \in \mathbb{Z}\}$ (c) $\bigcup_{A \in P(\mathbb{N})} A$

In (a) we have $(0, n] = \{x \in \mathbb{R} \ : \ 0 < x \leq n\}$, the real interval from 0 to n that does not contain 0 but contains n.

Solution.
(a) $\bigcup_{n \in \mathbb{N}} (0, n] = (0, 1] \cup (0, 2] \cup (0, 3] \cup \cdots = \{x \in \mathbb{R} \ : \ x > 0\}$

These sets are equal because every $x \in (0, n]$ for some $n \in \mathbb{N}$ is also in the set on the right hand side. Conversely let $x \in \mathbb{R}$ such that $x > 0$. Then there exists $n \in \mathbb{N}$ such that $x \in (0, n]$. Hence x is in the set on the left hand side.

(b) $\bigcap_{n=1}^{3} \{nz : z \in \mathbb{Z}\}$
$= \mathbb{Z} \cap \{\dots, -2, 0, 2, 4, 6 \dots\} \cap \{\dots, -3, 0, 3, 6, 9 \dots\}$
$= \{\dots, -6, 0, 6, 12, 18, \dots\}$
$= \{6z : z \in \mathbb{Z}\}$

(c) $\bigcup_{A\in P(\mathbb{N})} A = \emptyset \cup \{1\} \cup \{2\} \cup \cdots \cup \{1,2\} \cup \cdots = \mathbb{N}$

(7) Simplify the following sets and justify your answers:
(a) $\bigcap_{n\in\mathbb{N}}\{nz \ : \ z \in \mathbb{Z}\}$ (b) $\bigcup_{x\in\mathbb{R}}[-x,x]$ (c) $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n},\frac{1}{n})$
In (c) we have $(-\frac{1}{n},\frac{1}{n})$ the open interval not containing the end points.

Solution. (a) $\bigcap_{n\in\mathbb{N}}\{nz \ : \ z \in \mathbb{Z}\} = \{0\}$ because 0 is the only integer that is a multiple of every natural number.
(b) $\bigcup_{x\in\mathbb{R}}[-x,x] = \mathbb{R}$ because every real x is in some interval of the union, namely $[-|x|,|x|]$.
(c) $\bigcap_{n\in\mathbb{N}}(-\frac{1}{n},\frac{1}{n}) = \{0\}$ because 0 is the only real number that is contained in every interval $(-\frac{1}{n},\frac{1}{n})$ for $n \in \mathbb{N}$.