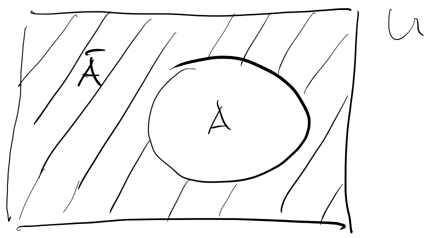
Subtraction Principle (Fact 3.3). For a subset A of a set U,

$$|A| + |\overline{A}| = |U|$$



Example. How many 3-letter words have a repeated letter? e.g. ada, bee

Repetitions may be in positions 1-2, 1-3, 2-3, 1-2-3;
difficult to count.

But indirectly

U = set of all 3-letter words $|U| = 26^3$

A = set of all 3-letter words without repetition

$|A| = 26 . 25 . 24$

$\bar{A}$ = with repetition

$|\bar{A}| = |U| - |A|$

Counting subsets.

Question. How many 3-element subsets are there in $\{a, b, c, d, e\}$?

$5 \cdot 4 \cdot 3 = \frac{5!}{2!}$ lists of 3 elements without repetition.

But these lists

$\boxed{a}\ \boxed{b}\ \boxed{c}$

$\boxed{b}\ \boxed{a}\ \boxed{c}$

represent the same set $\{a, b, c\}$

How many distinct lists yield the same set?

Every permutation of $\boxed{a}\ \boxed{b}\ \boxed{c}$ yield the same set.

$3!$ lists correspond to 1 set.

Divide the number of lists by the number of permutations of 3 elements:

$\underbrace{\frac{5!}{2!\ 3!}}_{=5}$ subsets of size 3.

Theorem. *For $k, n \in \mathbb{N}$, $k \leq n$, the number of k-element subsets of an n-element set is*

$$\frac{n!}{k!\,(n-k)!} =: \binom{n}{k} \qquad \text{binomial coefficient “}n\text{ choose }k\text{”}$$

Proof. The number of k-tuples without repetition and entries from $\{1, \ldots, n\}$ is $n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$

All $k!$ permutations of a k-tuple yield the same subset.

So $\#$ k-subsets $= \frac{\#\ k\text{-tuples without repetition}}{\#\ \text{permutations}} = \frac{n!}{(n-k)!\ k!}$ $\square$

Example. 1) How many distinct 5-card hands are there in Poker (52 cards in total)?

$\binom{52}{5} = \frac{52!}{47!\ 5!} = \frac{52 \cdots 48}{5!} = 2.6$ million

2) Hands with 3 of a kind?

$$\underset{\substack{\uparrow \\ \text{kind}}}{\binom{13}{1}} \cdot \underset{\substack{\uparrow \\ \text{suits}}}{\binom{4}{3}} \cdot \binom{12}{2} \cdot \binom{4}{1}^2 = 54\,912$$

Theorem. *For integers $1 \leq k \leq n$,*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof. $\#$ k-subsets of $\{1, \ldots, n+1\}$ =

= $\#$ k subsets not containing $n+1$ + $\#$ k-subsets containing $n+1$

= $\#$ k-subsets of $\{1, \ldots, n\}$ + $\#$ $(k-1)$-subsets of $\{1, \ldots, n\}$ $\square$

Pascal's triangle.

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
\qquad
\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3}
\end{array}$$

Binomial Theorem. For $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k .$$

Combinatorial proof.

$$(x+y)^n = \underbrace{(x+y) \cdots (x+y)}_{n \text{ times}}$$

Expand and write the summands as products of length n.

E.g. $(x+y)^2 = xx + \underline{xy} + \underline{yx} + yy$

$(x+y)^3 = xxx + \underline{xyx} + \underline{yxx} + yyx$
$+ \underline{xxy} + xyy + yxy + yyy$

Every product with $n-k$ x and k y yields $x^{n-k} \cdot y^k$.

There are $\binom{n}{k}$ such products (k positions out of n to choose for y)

Hence the coefficient of $x^{n-k} y^k$ is $\binom{n}{k}$. $\square$