ZERMELO-FRAENKEL SET THEORY WITH THE AXIOM OF CHOICE (ZFC)

PETER MAYR (MATH 2001, CU BOULDER)

Around 1900 Zermelo and Fraenkel proposed axioms to formalize the notion of sets and to avoid paradoxes like Russell's paradox. They are now the most common foundation of mathematics. We describe them in a somewhat informal way.

Equality.

(1) (Extensionality) Sets A and B are equal if they have the same elements.

Existence of an infinite set.

(2) (Infinity) There exists a set that has infinitely many elements (e.g. $\mathbb{N}$).

Creating new sets.

(3) (Specification) For a set A and a property P given by a formula,
$$\{x \in A \ : \ x \text{ satisfies } P\}$$
is a set.

(4) (Replacement) For a set A and a function f,
$$f(A) := \{f(x) \ : \ x \in A\}$$
is a set.

(5) (Power set) For every set A, there exists the set of all its subsets
$$P(A) := \{B \ : \ B \subseteq A\}.$$

(6) (Pairing) For any two sets A and B, there exists the set $\{A, B\}$.

(7) (Union) For a set I and sets A_i for $i \in I$,
$$\bigcup_{i \in I} A_i := \{x \ : \ x \in A_i \text{ for some } i \in I\}$$
is a set.

(8) (Choice) For a set I and non-empty disjoint sets A_i for $i \in I$, there exists a set C that intersects each A_i in exactly one element.

No set is an element of itself.

(9) (Regularity) Every non-empty set A has an element B such that A and B are disjoint.

References

[1] Paul Halmos. Naive Set Theory. Springer Verlag, New York, 1974.

Date: August 26, 2025.