# The structure of idempotent involutive residuated lattices and weakening relation algebras

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January 15, 2020

Joint AMS-MAA Mathematics Meeting, Denver

# Overview

Part I, with O. Tuyt and D. Valota

- Commutative idempotent involutive residuated lattices
- Gluing construction
- Ungluing decomposition

Part II, with N. Galatos

- $\mathrm{FL}^2$-algebras and their congruences
- Weakening relation algebras
- Double-division conuclei

# Involutive residuated lattices

## Definition

A **pointed residuated lattice** $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1, 0\rangle$ is

- a lattice $\langle A, \wedge, \vee\rangle$ and a monoid $\langle A, \cdot, 1\rangle$ such that

$$x \cdot y \leq z \iff x \leq z/y \iff y \leq x\backslash z \qquad \text{for all } x, y, z \in A.$$

$\mathbf{A}$ is **involutive** if ${\sim}{-}x = x = -{\sim}x$, where ${\sim}x = x\backslash 0$ and $-x = 0/x$.

$\backslash, /$ can be term-defined via $x\backslash y = {\sim}(-y \cdot x)$ and $x/y = -(y \cdot {\sim}x)$.

- $\mathbf{A}$ is **commutative** if $x \cdot y = y \cdot x$ (hence $-x = {\sim}x$)
- $\mathbf{A}$ is **idempotent** if $x \cdot x = x$ for all $x \in A$

CIdInRL denotes the variety of **commutative idempotent involutive residuated lattices**.

# Examples of CIdINRLs

Let $\mathbf{A} \in$ CIdInRL.

- $\langle A, \cdot, 1 \rangle$ is a meet-semilattice with top element 1 and order $\sqsubseteq$ (**monoidal order**) defined as

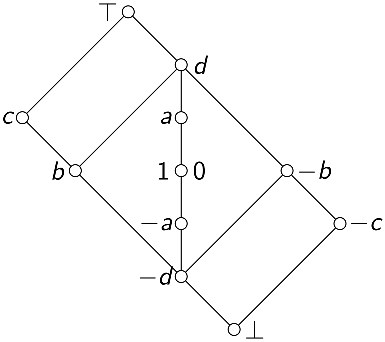
$$a \sqsubseteq b \iff a \cdot b = a.$$

Hence, the orders $\leq$ and $\sqsubseteq$ together with the involution $-$ completely determine $\mathbf{A}$, allowing us to work in the signature $\langle A, \vee, \cdot, -, 0, 1 \rangle$

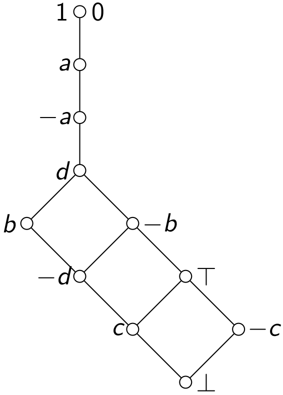
- **Boolean algebras** (where $\leq = \sqsubseteq$)
- **Sugihara monoids** defined as **distributive** CIdInRLs (= algebraic semantics for relevance logic $\mathrm{RM}^{\mathrm{t}}$)

  Dunn [1970] proved that the subdirectly irreducible Sugihara monoids are linearly ordered. Up to isomorphism, there is one such algebra $\mathbf{S}_n$ for each chain with $n$ elements.

# Another example




$\langle A, \leq \rangle$




$\langle A, \sqsubseteq \rangle$

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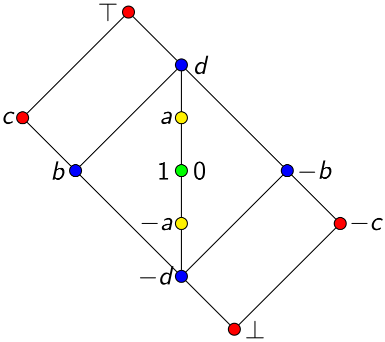
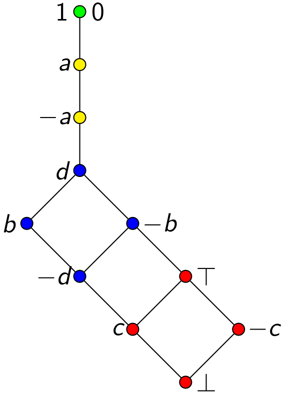



$\langle A, \leq \rangle$




$\langle A, \sqsubseteq \rangle$

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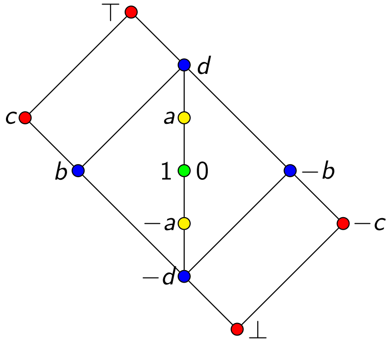
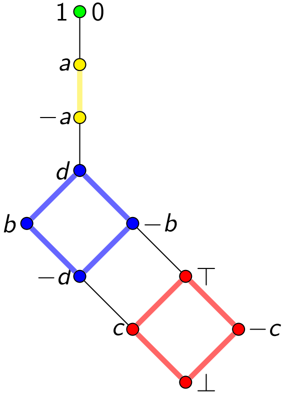



$\langle A, \leq \rangle$




$\langle A, \sqsubseteq \rangle$

# Some properties

For each $x \in A$, let

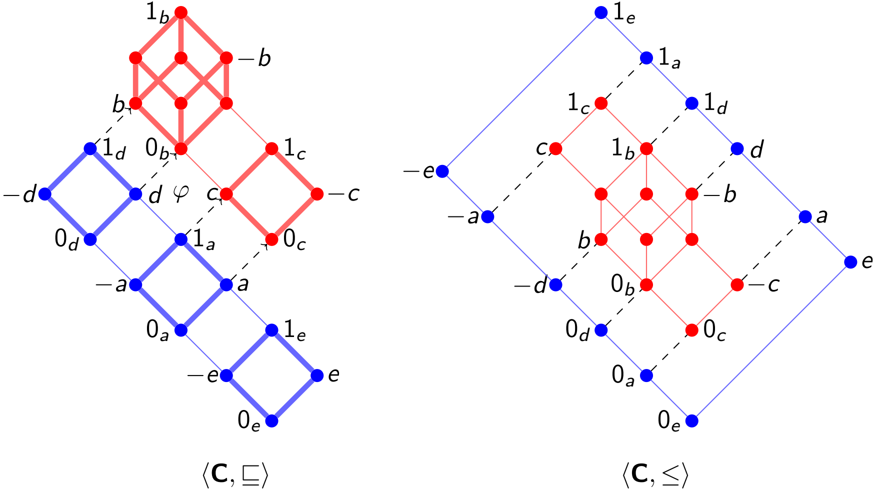
$$\begin{aligned} 0_x &:= x \wedge -x = x \cdot -x \\ 1_x &:= x \vee -x = -(x \cdot -x) = x/x \\ \mathbb{B}_x &:= \{y \in A \mid 0_x \sqsubseteq y \sqsubseteq 1_x\} \\ {\downarrow}0 &:= \{y \in A \mid y \leq 0\} = \{0_x \mid x \in A\} \end{aligned}$$

## Lemma

- *For each $x \in A$, $\langle \mathbb{B}_x, \wedge, \vee, -, 0_x, 1_x \rangle$ is a* **Boolean algebra**
- *For each $x \in A$, the monoidal order and the lattice order* **agree** *on $\mathbb{B}_x$*
- *The monoidal intervals $\mathbb{B}_x$* **partition** *$A$*
- *$\langle {\downarrow}0, \cdot, \vee \rangle$ is a* **distributive lattice** *with top element $0$*

*Hence, the monoidal semilattice is a disjoint union of Boolean algebras over the 'skeleton' of a distributive lattice.*

# Construction: example of $\mathbf{C} = \mathbf{A} \oplus_\phi \mathbf{B}$



# Construction: formally

Let $\Uparrow a = \{x \in A \mid a \sqsubseteq x\}$ and $\Downarrow b = \{x \in B \mid x \sqsubseteq b\}$.

$\mathbf{A} = \langle A, \vee^A, \cdot^A, -^A, 0^A, 1^A \rangle$ (the bottom algebra) and $\mathbf{B} = \langle B, \vee^B, \cdot^B, -^B, 0^B, 1^B \rangle$ (the top algebra) are **$\varphi$-compatible** if

- $\varphi$ is a **bijection** $\Uparrow a \to \Downarrow b$ for some $a \leq 1^A$ and $0^B \leq b \leq 1^B$ such that
- $\varphi$ **preserves join**, i.e. $\varphi(x \vee^A y) = \varphi(x) \vee^B \varphi(y)$
- $\varphi$ **preserves fusion**, i.e. $\varphi(x \cdot^A y) = \varphi(x) \cdot^B \varphi(y)$ and
- $0^B = \varphi(a \vee^A 0^A)$.

For $\varphi$-compatible algebras we define a **glueing construction** $\oplus_\varphi$

# Glueing construction

$$\mathbf{A} \oplus_\varphi \mathbf{B} \coloneqq \langle A \uplus B, \vee, \cdot, -, 1^{\mathbf{B}}, 0^{\mathbf{B}} \rangle$$

$$x \vee y = \begin{cases} x \vee^A y & \text{if } x, y \in A \\ x \vee^B y & \text{if } x, y \in B \\ \varphi(x \vee^A a) \vee^B y & \text{if } x \in A,\ y \in B,\ x \leq^A -^A a \\ x \vee^A \varphi^{-1}(y \cdot^B b) & \text{if } x \in A,\ y \in B,\ x \not\leq^A -^A a \end{cases}$$

$$x \cdot y = \begin{cases} x \cdot^A y & \text{if } x, y \in A \\ x \cdot^B y & \text{if } x, y \in B \\ x \cdot^A \varphi^{-1}(y \cdot^B b) & \text{if } x \in A,\ y \in B \end{cases}$$

$$-x = \begin{cases} -^A x & \text{if } x \in A \\ -^B x & \text{if } x \in B \end{cases}$$

# Glueing construction

**Theorem**

*For* $\varphi$-**compatible** $\mathbf{A}, \mathbf{B} \in \mathsf{CIdInRL}$ *the algebra* $\mathbf{A} \oplus_\varphi \mathbf{B}$ *is in* $\mathsf{CIdInRL}$.

The proof is by case analysis and direct computation.

# Unglueing decomposition

For finite $\mathbf{C} \in \mathsf{CIdInRL}$, consider a co-atom $c$ in the underlying distributive lattice with universe $\downarrow 0 = \{0_x \mid x \in C\}$.

By distributivity, there exists $c^*$ such that $\langle c, c^* \rangle$ is a splitting pair of $\downarrow 0$.

Note: $c = 0_c$, hence $-c = 1_c$.

**Lemma**

*The pair $\langle 1_c, c^* \rangle$ is a splitting pair of $(C, \sqsubseteq)$.*

*Moreover, $\Uparrow c^*$ is a subuniverse of $\mathbf{C}$, and $\Downarrow 1_c$ is closed under $\vee, \cdot, -$*

# Unglueing decomposition

Let $\mathbf{A} = \langle \Downarrow 1_c, \vee, \cdot, -, 1_c, 0_c \rangle$.

Let $\mathbf{B}$ be the subalgebra of $\mathbf{C}$ with subuniverse $\Uparrow c^*$.

Choose $a = 1_c \cdot c^*$ and $b = (1_c \vee -a) \vee c^*$, and define

$\varphi(x) = (x \wedge -a) \vee c^*$ for $a \sqsubseteq x \sqsubseteq 1_c$.

## Lemma

- *$a \leq 1_c$ and $0 \leq b \leq 1$*
- *$\varphi$ is a bijection to $\{y \mid c^* \sqsubseteq y \sqsubseteq b\}$ with $\varphi^{-1}(y) = y \cdot 1_c$*
- $\varphi(c \vee a) = 0_b$

## Theorem

*The algebra $\mathbf{C} \in \mathsf{CIdInRL}$ is isomorphic to $\mathbf{A} \oplus_\varphi \mathbf{B}$.*

# Structural characterization

The discovery of the previous theorem and the results below were guided by Prover9/Mace4 computations of all CIdInRLs with $\leq 16$ elements.

**Theorem**

*Any finite member* $\mathbf{A}$ *of* CIdInRL *can be constructed using the gluing construction, starting from finite Boolean algebras.*

**Corollary**

*Any finite* $\mathbf{A} \in$ CIdInRL *is determined by its fusion semilattice and also by its lattice reduct.*

To do: Implement an algorithm for constructing all finite CIdInRLs.

# Fusion-distributivity

As an application, call an $\mathbf{A} \in \mathsf{CIdInRL}$ fusion-distributive if the meet-semilattice $\langle A, \cdot \rangle$ is **distributive**, i.e. if for all $x, y, z \in A$,

$$x \cdot y \sqsubseteq z \implies \exists x', y' \in A \text{ such that } x \sqsubseteq x', y \sqsubseteq y', \text{ and } z = x' \cdot y'.$$

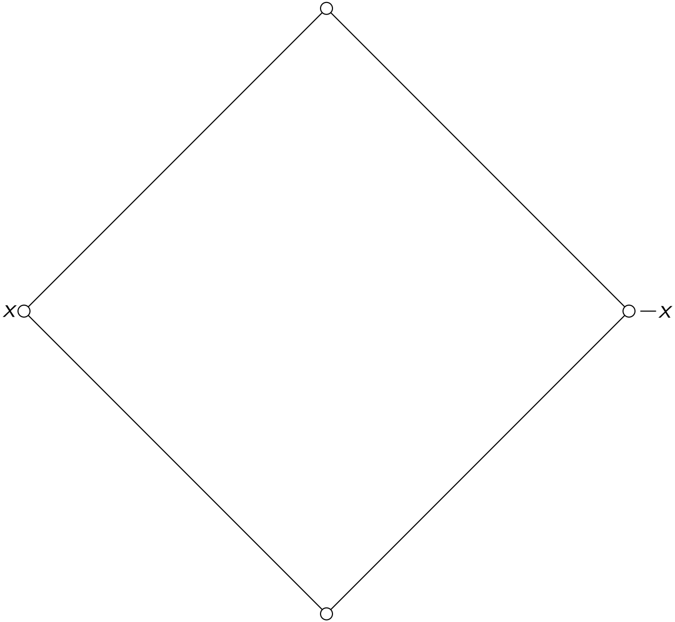
## Lemma

*For compatible fusion-distributive* $\mathbf{A}, \mathbf{B} \in \mathsf{CIdInRL}$*, their gluing* $\mathbf{C}$ *is fusion-distributive.*

## Corollary

- *Any finite* $\mathbf{A} \in \mathsf{CIdInRL}$ *is fusion-distributive.*
- *Every finite distributive lattice can occur as skeleton.*

# A one-generated infinite CIdInRL



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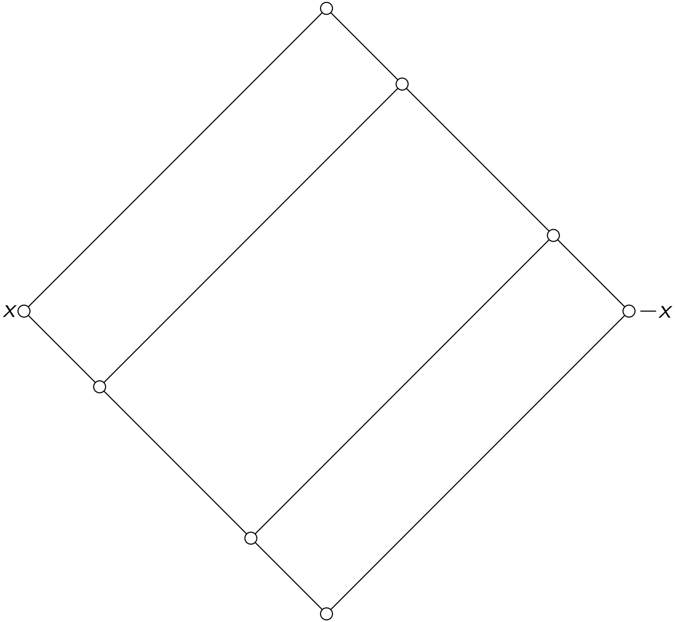

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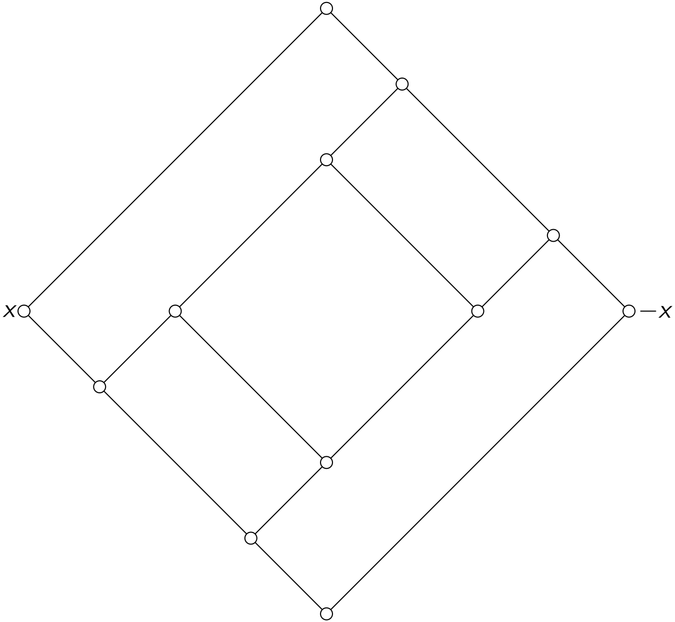

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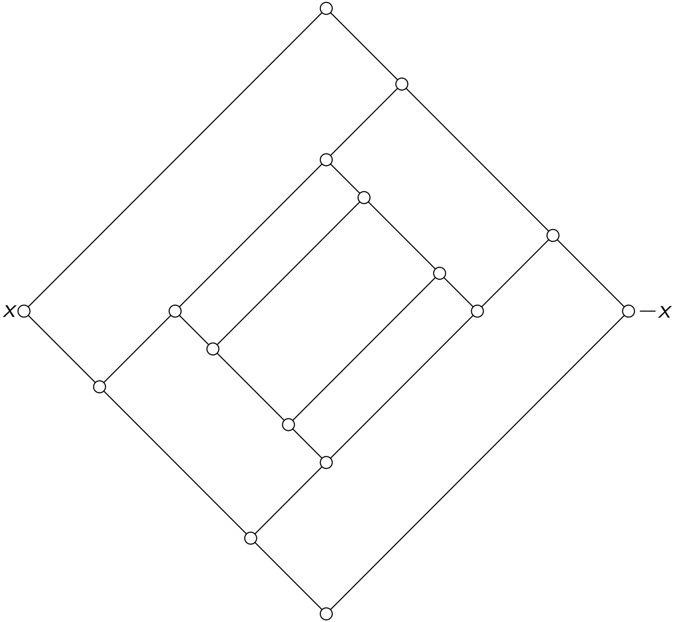

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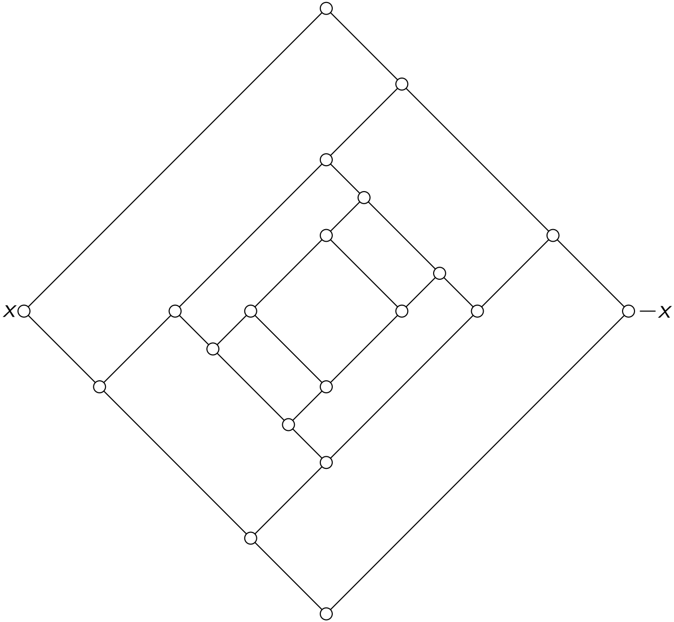

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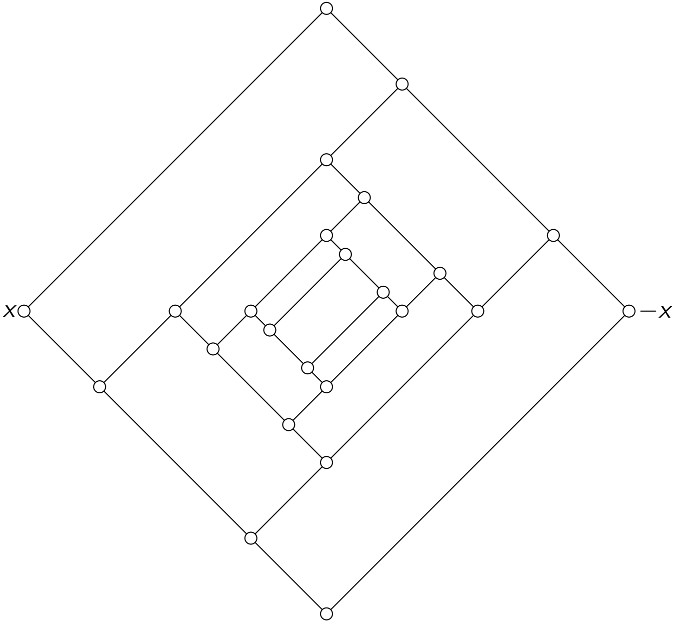

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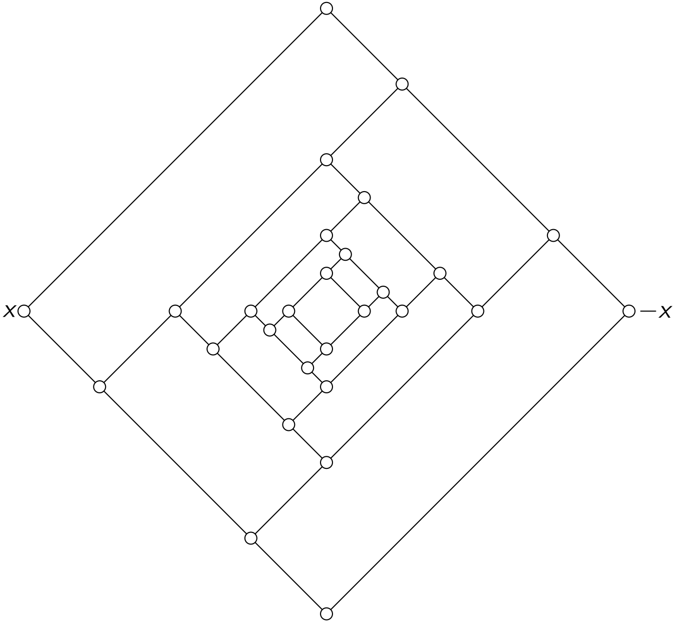

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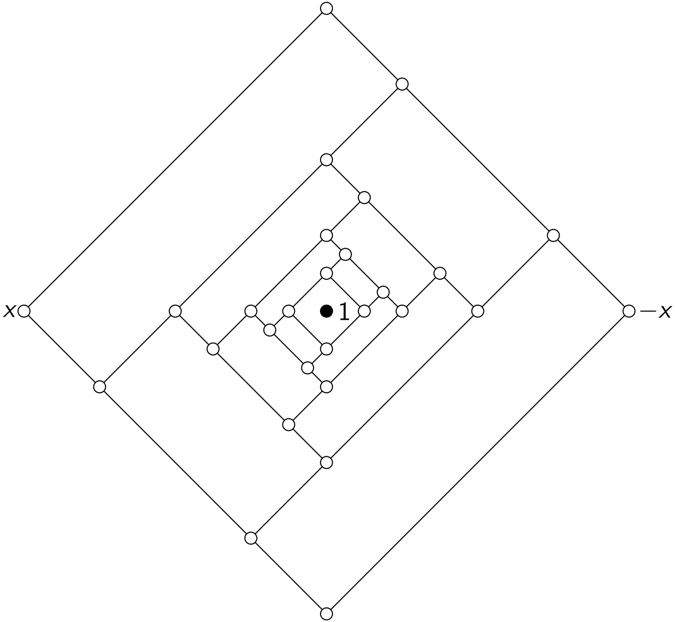

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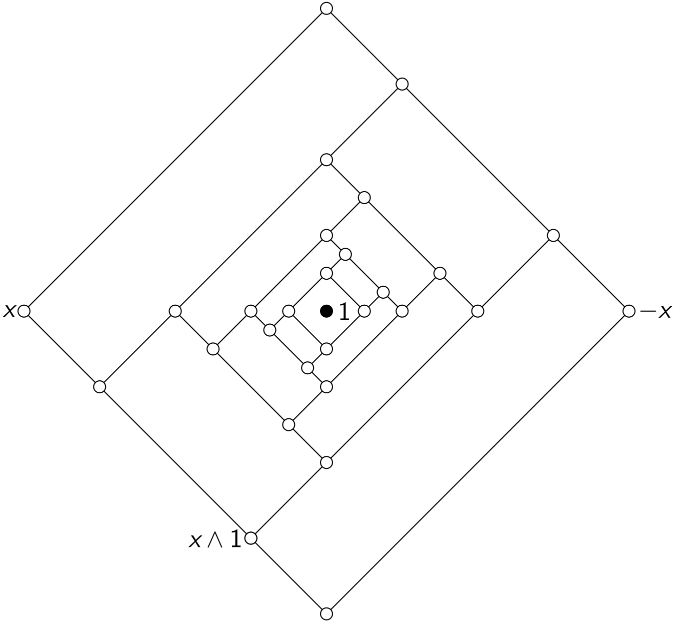

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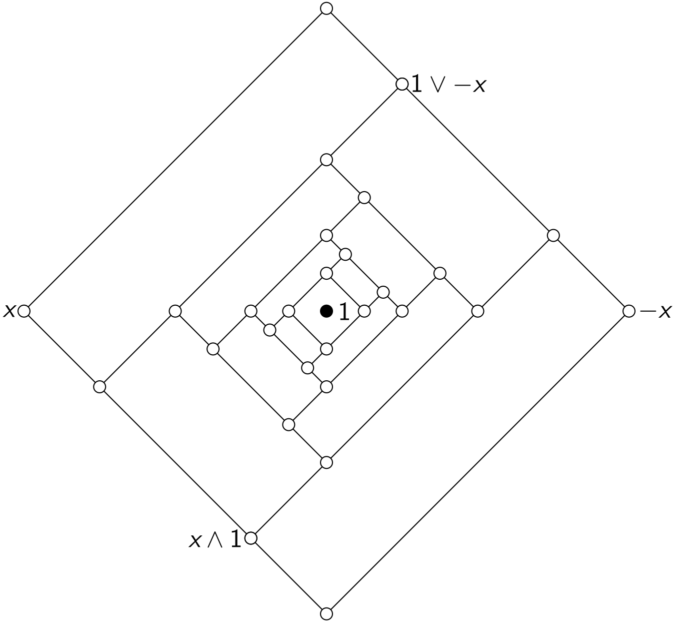

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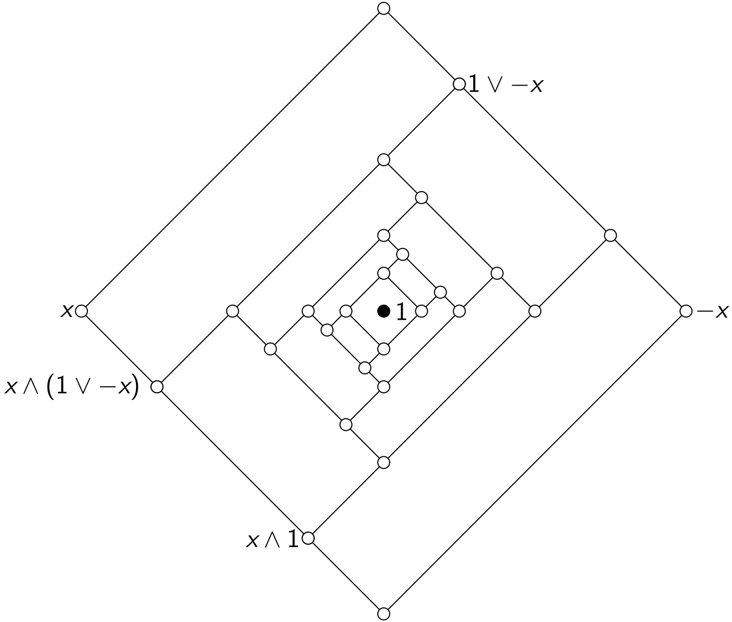

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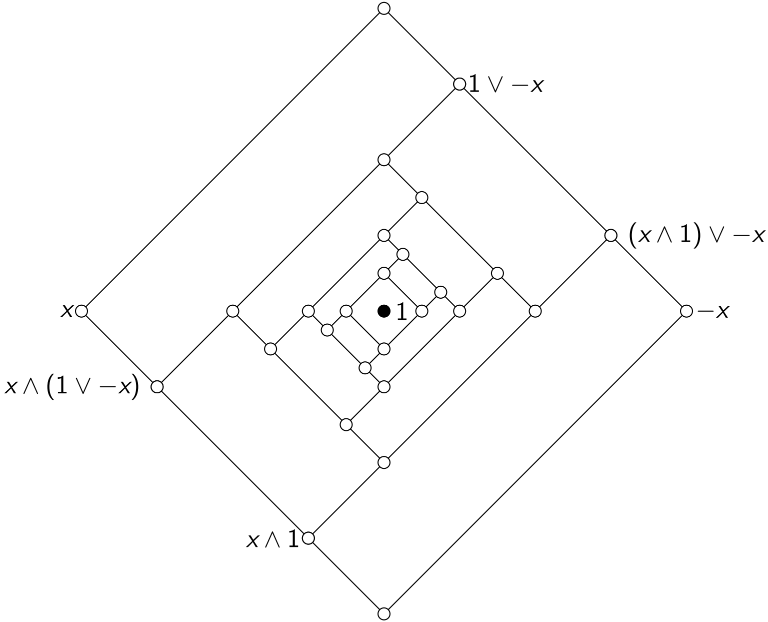
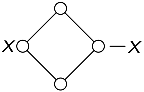


# The fusion semilattice of a one-generated infinite CIdInRL



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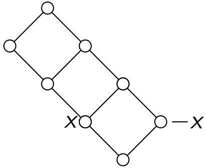

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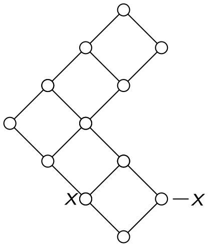

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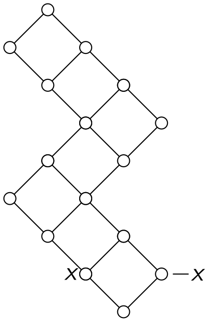

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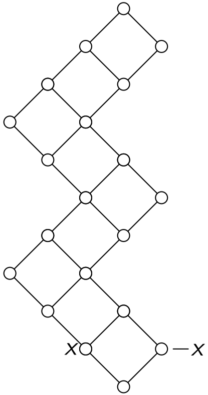

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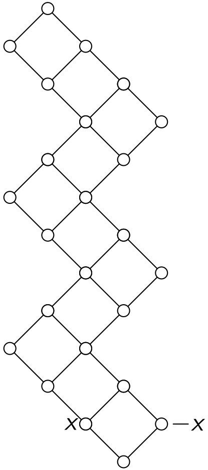

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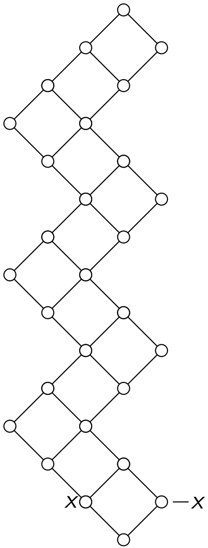

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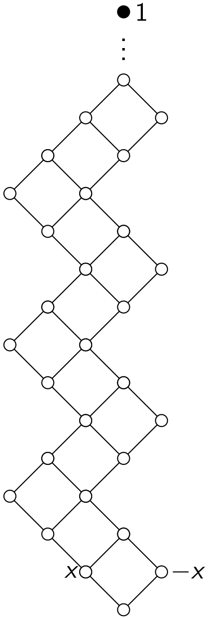

# Part II: $\mathrm{FL}^2$-algebras and GBI-algebras

A **$\mathbf{FL}^2$-algebra** is of the form $\mathbf{A} = (A, \wedge, \vee, \diamond, \rightarrow, \leftarrow, t, f, \cdot, \backslash, /, 1, 0)$ s. t.

$$\mathbf{A}_t = (A, \wedge, \vee, \diamond, \rightarrow, \leftarrow, t, f) \text{ and } \mathbf{A}_1 = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$$

are pointed residuated lattices.

**Relation algebras** are examples of **classical** $\mathrm{FL}^2$-algebras: $\mathbf{A}_t$ is a Boolean algebra with $x \wedge y = x \diamond y$.

A **bounded generalized bunched implication algebra** (**bGBI-algebra**) is a $\mathrm{FL}^2$-algebra that satisfies $x \wedge y = x \diamond y$, $t = \top$, $f = \bot$ and $0 = 1$.

A **bunched implication algebra**, or **BI-algebra**, is a commutative bGBI-algebra (i.e., $xy = yx$).

# Congruences of residuated lattices

A **congruence filter** of a residuated lattice $\mathbf{A}$ is a subset of the form $F = {\uparrow}([1]_\theta)$ where $\theta$ is a congruence.

Congruence filters satisfy the following **normality condition** for $a \in A$ (where quantifiers range over $F$):

$$\forall x \in F\ \exists x_1, x_2 \in F,\ x_1 a \leq ax \text{ and } ax_2 \leq xa. \qquad (N_a)$$

A filter $F$ satisfies $(N)$ if $(N_a)$ holds for all $a \in A$.

The set of **congruence filters** of $\mathbf{A}$ is denoted by $\mathrm{CF}(\mathbf{A})$.

**Theorem (Blount-Tsinakis 2003)**

*For a residuated lattice $\mathbf{A}$, a subset $F$ is a congruence-filter if and only if $F$ is a lattice filter and a submonoid of $\mathbf{A}$ that satisfies $(N)$.*
*Moreover, $\mathrm{Con}(\mathbf{A})$ is isomorphic to the lattice $\mathrm{CF}(\mathbf{A})$ of congruence-filters via the bijection $\theta \mapsto {\uparrow}([1]_\theta)$ and $F \mapsto \{(x, y) : x/y, y/x \in F\}$.*

# Congruences of $\mathsf{FL}^2$-algebras

For $\mathsf{FL}^2$ the congruence 1-filters are determined by a stronger ***t*-normality** condition. For any $a \in A$

$$\forall x \in F,\ \exists x_1, x_2, x_3, x_4 \in F, \qquad (N_a^t)$$
$$ax_1 \le a \diamond xt, \quad x_2 a \le xt \diamond a, \quad a \diamond x_3 t \le xa, \quad x_4 t \diamond a \le ax$$

A filter $F$ satisfies $(N^t)$ if $(N_a^t)$ holds for all $a \in A$.

**Theorem**

*For an $\mathsf{FL}^2$-algebra $\mathbf{A}$, a subset $F$ is the 1-filter of some congruence $\theta$ of $\mathbf{A}$ if and only if $F$ is a lattice filter and $\cdot, 1$-submonoid of $\mathbf{A}$ that satisfies $(N^t)$*

An analogous result holds for congruence $t$-filters $\uparrow([t]_\theta$ of $\mathsf{FL}^2$-algebras.

# Congruences of GBI-algebras

The previous result specializes to generalized bunched implication algebras:

## Corollary

*The 1-filters of a GBI-algebra* $\mathbf{A}$ *are the filter submonoids that are closed under the terms*

$$u_a(x) = a\backslash(a \wedge x\top),\ \ v_a(x) = (a \to xa)/\top \text{ and } \rho_a(x) = ax/a,$$

A previously known characterization of the congruence classes of GBI-algebras used more complicated terms with two parameters.

Similar 1-parameter terms exist for congruence $\top$-filters of GBI-algebra.

## Theorem

*For an involutive GBI-algebra, a lattice filter* $F$ *is a* $\top$*-filter if and only if for all* $x \in F$ *it follows that* $\neg{\sim}x, \quad \neg{-}x, \quad {\sim}(\top(-x)\top) \in F$.

# Weakening relation algebras

For a poset $\mathbf{P} = (P, \leq)$, let $\mathrm{Wk}(\mathbf{P}) = \{R \subseteq P^2 : \leq;R;\leq \subseteq R\}$.

Relations in $\mathrm{Wk}(\mathbf{P})$ are called **weakening closed relations** since

$$x \leq u \; R \; v \leq y \implies x \; R \; y$$

$\sim R := (R^c)^\smile = \{(y, x) \mid (x, y) \notin R\}$, the **complement-converse** of $R$.

Weakening relations are closed under **complement-converse**, **union**, **intersection**, Heyting **implication** $\to$ ($=$ residual of intersection), relation **composition** ; and **residuals** $\backslash, /$ of composition.

$1 := \leq$ is a weakening relation and is the identity of composition.

The **full weakening relation algebra** on a poset $\mathbf{P}$ is

$$\mathbf{Wk}(\mathbf{P}) = (\mathrm{Wk}(\mathbf{P}), \cap, \cup, \to, P^2, \emptyset, ;, \sim, 1, 0), \text{ where } 0 = \sim 1.$$

Representable weakening relation algebras $= \mathrm{V}\{\mathbf{Wk}(\mathbf{P}) \mid \mathbf{P} \text{ is a poset}\}$.

# Double division conuclei

An **interior operator** $\delta$ on a poset is an order-preserving map such that $\delta(\delta(x)) = \delta(x) \leq x$.

An interior operator $\delta$ is a **conucleus** if $\delta(x)\delta(y) \leq \delta(xy)$.

The conucleus **image** $\delta(\mathbf{A})$ of a residuated lattice is a residuated lattice $(\delta(A), \wedge_\delta, \vee, \cdot, \backslash_\delta, /_\delta)$ without 1, where $x *_\delta y = \delta(x * y)$ for $* \in \{\wedge, \backslash, /\}$.

Let $p \in A$ be a **positive idempotent**, i.e., $p = p^2 \geq 1$.

Then $\delta_p(x) = p\backslash x/p$ is a conucleus called the **double division conucleus**.

**Lemma**

*$\delta_p(\mathbf{A}) = \{pxp \mid x \in A\}$, and $p$ is the identity element.*

# Double division conuclei of relation algebras

In a full relation algebra, a positive idempotent $p$ is a **preorder** $\mathbf{P} = (P, \sqsubseteq)$ (i.e., $p = \sqsubseteq$ is reflexive and transitive).

If $p \wedge p^{\smile} = 1$ then $\mathbf{P}$ is a poset and $\mathbf{Wk}(\mathbf{P}) = \delta_p(\mathrm{Rel}(P))$.

Hence the variety RWkRA of representable weakening relation algebras contains all double division conucleus images of members of RRA.

For a class $\mathcal{K}$ of algebras let $\mathrm{d}\mathcal{K} = \{\delta_p(\mathbf{A}) : \mathbf{A} \in \mathcal{K}, 1 \leq p^2 = p \in A\}$.

**Theorem**

*If $\mathcal{V}$ is a variety of bounded GBI-algebras with $\top\backslash x/\top$ as unary discriminator on the subdirectly irreducible members then $S(\mathrm{d}\mathcal{V})$ is a discriminator variety with the same unary discriminator term.*

Applying this result to the variety RA produces the discriminator variety $S(\mathrm{dRA})$ that contains both RA and RWkRA.

# Some identities that hold in $S(\mathrm{dRA})$

Recall that the variety RA of **relation algebras** is an abstract counterpart of the variety RRA of **representable relation algebras**.

The variety $S(\mathrm{dRA})$ generated by double-division conucleus images of relation algebras is the abstract counterpart of RWkRA.

**Open problem**: Find a (finite?) axiomatization of $S(\mathrm{dRA})$.

In a GBI-algebra let the **domain** $d(x) = x\top \wedge 1$ and **range** $r(x) = \top x \wedge 1$.

**Theorem**

*The identities*

$$d(x)x = x, \quad xr(x) = x, \quad \top x \top x \top = \top x \top \;\; \textit{and} \;\; {\sim}\neg(xy) \leq ({\sim}\neg y)({\sim}\neg x)$$

*hold in* $S(\mathrm{dRA})$.

# References


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Thank you!